

APPROXIMATING MULTIVARIATE TEMPERED STABLE PROCESSES

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Abstract

We give a simple method to approximate multidimensional exponentially tempered stable processes and show that the approximating process converges in the Skorokhod topology to the tempered process. The approximation is based on the generation of a random angle and a random variable with a lower dimensional Lévy measure. We then show that if an arbitrarily small normal random variable is added, the marginal densities converge uniformly at an almost linear rate, providing a critical tool to assess the performance of existing particle tracking codes.

Keywords: Tempered stable process, Simulation, Lévy process, Operator stable process, Spectral measure

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1. Introduction

Being able to approximate stable and tempered stable processes is important for investigation and simulation purposes. A stable or tempered stable process is a process whose increments are independent stable or tempered stable random vectors. An exponentially tempered stable random vector X_ρ is obtained from a stable random vector X by exponentially cooling its jump size (or Lévy measure); the general class of tempered stable vectors was introduced by [15]. Tempered stable laws are used in physics as a model for turbulent velocity fluctuations [11, 14], as well as in finance [3, 4] and hydrology [13, 16] as a model of transient anomalous diffusion [1]. As the random

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variables are infinitely divisible, they can be approximated using LePage's method in splitting up the integral in their Lévy Khintchine representation into a compound Poissonian part (with tempered Pareto jumps) and an approximately normal part [8]. For general processes with operator scaling this is also nicely exhibited in [7]. In case of a stable random vector this involves adding a random number of random vectors comprised of one-dimensional Pareto jumps multiplied by a random direction drawn according to the mixing measure M . In Zhang et al. [17] the authors compared this approach with just drawing a random direction multiplied by a one-dimensional skewed stable variable which can also easily be generated [6]. As they are in the same domain of attraction of an operator stable [12], both approaches work well. However in the case of tempered stable vectors, they are in the domain of attraction of the multivariate normal and hence another argument is needed. Furthermore, the rate of convergence was not explored; this article is addressing these issues.

In this paper we explore approximations to a stochastic process with stationary, independent increments for which we are given a coordinate transform $T : \Omega_\theta \times \Omega_r \rightarrow \mathbb{R}^d$ for some measure spaces Ω_θ and Ω_r which decomposes the Lévy measure ϕ of the stochastic process into lower dimensional Lévy measures; i.e., assume that we have a probability measure M on Ω_θ and for each $\theta \in \Omega_\theta$ there is a measure ϕ_θ on Ω_r such that, with I_A being the indicator function of a set A ,

$$\begin{aligned} \int I_A(x)\phi(dx) &= \int_{\Omega_\theta \times \Omega_r} I_A(T(\theta, r))\phi(dT(\theta, r)) \\ &= \int_{\Omega_\theta} \int_{\Omega_r} I_A(T(\theta, r))\phi_\theta(dr)M(d\theta) \end{aligned} \quad (1)$$

for any measurable set A and the induced degenerate measure on \mathbb{R}^d via

$$\tilde{\phi}_\theta(A) := \int_{\Omega_r} I_A(T(\theta, r))\phi_\theta(dr) \quad (2)$$

is also Lévy. Note that for most processes appearing in applications there is a canonical decomposition.

We show that for $\tau \rightarrow 0$ the processes

$$S_\tau(t) = \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} X_{\Theta_j}^j(\tau),$$

converge to the original stochastic process in the Skorokhod topology, where the increments $X_{\Theta_j}^j(\tau)$ are distributed according to the infinitely divisible distributions

with Lévy measure $\tau_{\tilde{\phi}_{\Theta^j}}$ and Θ^j are random vectors distributed according to M . In case that $\Omega_r \subset \mathbb{R}$ is a set of scale factors (radii), $\Omega_\theta \subset \mathbb{R}^d$ is a $d - 1$ -dimensional hypersurface (directions) and $T(\theta, r) = \theta r$, the increments take the simpler form $X_{\Theta^j}^j(\tau) = \Theta^j r_{\Theta^j}$, where r_{Θ^j} is a random variable with Lévy measure $\tau_{\phi_{\Theta^j}}$ and Θ^j is a random direction picked according to M .

In particular, if $\Omega_\theta = \{\theta : \|\theta\| = 1\}$, $\Omega_r = \mathbb{R}$ and if ϕ_θ is the Lévy measure of a one-dimensional tempered stable, this provides a method to approximate multidimensional tempered stable processes, extending the one-dimensional method to obtain tempered stable laws as random walk limits developed recently by Chakrabarty and Meerschaert [5].

We then investigate the *rate of convergence* in the exponentially tempered stable case with a single α [15] and show that the densities at a fixed time t converge in the L^2 -multiplier norm a rate of $C(1 + \log^2 n)/n$. We further show that given a small random initial perturbation, the marginal densities of the processes converge uniformly in x and t at an almost linear rate. This is important as it provides a performance measure/benchmark for existing particle tracking codes [16], where this approximation method was used to estimate the density of the tempered stable model (without knowing how accurate the density approximation was), comparing it to a numerical simulation of particle transport in a random fracture network.

Furthermore, we also show that if the Lévy measure can be decomposed into finitely many tempered stable measures (e.g. of different α 's) then for each step randomly choosing which tempered α -stable to simulate also converges at an almost linear rate.

2. The general result

Let $X(t)$ be a stochastic process in \mathbb{R}^d with stationary, independent increments. Then $X(t)$ has a unique Lévy-Khintchine representation

$$E[\exp(-i\langle k, X(t) \rangle)] = \exp(t\psi(k))$$

and log-characteristic function

$$\psi(k) = i\langle k, v \rangle - \frac{1}{2}\langle k, Qk \rangle + \int_{\mathbb{R}^d} \left(e^{-i\langle k, x \rangle} - 1 + \frac{i\langle k, x \rangle}{1 + \|x\|^2} \right) \phi(dx)$$

for some drift vector v , covariance matrix Q and measure ϕ satisfying

$$\int \frac{\|x\|^2}{1 + \|x\|^2} \phi(dx) < \infty.$$

Such a measure is called a Lévy measure.

Definition 1. Assume for a Lévy measure ϕ there exist measure spaces Ω_θ, Ω_r and a measurable map $T : \Omega_\theta \times \Omega_r \rightarrow \mathbb{R}^d$ such that

$$\int I_A(x) \phi(dx) = \int_{\Omega_\theta} \int_{\Omega_r} I_A(T(\theta, r)) \phi_\theta(dr) M(d\theta) \quad (3)$$

for any measurable set A , and assume that the induced degenerate measure $\tilde{\phi}_\theta$ defined via $\tilde{\phi}_\theta(A) = \int_{\Omega_r} I_A(T(\theta, r)) \phi_\theta(dr)$ is Lévy. We then call (ϕ_θ, M) a *Lévy decomposition* of ϕ and we call $\tilde{\phi}_\theta$ the projected Lévy measure in the direction of θ .

Let Θ be a Ω_θ -valued random variable with $\Pr\{\Theta \in A\} = M(A)$. Let $X_\theta(\tau)$ be a random variable with characteristic function

$$E[\exp(-i\langle k, X_\theta(\tau) \rangle)] = \exp(\tau \psi_\theta(k))$$

and

$$\psi_\theta(k) = i\langle k, v \rangle - \frac{1}{2}\langle k, Qk \rangle + \int_{\mathbb{R}^d} \left(e^{-i\langle k, x \rangle} - 1 + \frac{i\langle k, x \rangle}{1 + \|x\|^2} \right) \tilde{\phi}_\theta(dx).$$

Clearly, $\psi(k) = \int_{\Omega_\theta} \psi_\theta(k) M(d\theta)$. Let $X_\theta^j(\tau)$ and Θ^j , $j \in \mathbb{N}$, $\tau > 0$, be random variables on the same probability space distributed as $X_\theta(\tau)$ and Θ , respectively, all independent, and define the approximate process

$$S_\tau(t) = \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} X_{\Theta^j}^j(\tau),$$

where $\lfloor \frac{t}{\tau} \rfloor$ denotes the integer part of $\frac{t}{\tau}$. We are ready to state the first theorem.

Theorem 1. *Let X be a stochastic process in \mathbb{R}^d with stationary, independent increments and a Lévy measure with Lévy decomposition (ϕ_θ, M) . If*

$$\int_{\Omega_\theta} |\exp(\tau |\psi_\theta(k)|)| M(d\theta) < \infty$$

for some $\tau > 0$ and all $k \in \mathbb{R}^d$, then $S_\tau \rightarrow X$ in the Skorokhod topology as $\tau \rightarrow 0+$.

Proof. By design S_τ has independent increments. According to [9, Corollary VII.4.43] all we have to show is that the characteristic function of $S_\tau(t)$ converges to the characteristic function $\exp(t\psi(k))$ of $X(t)$ uniformly on compact intervals in t for all $k \in \mathbb{R}^d$.

Conditioning on Θ , the characteristic function of $S_\tau(t)$ is given by

$$E[e^{-i\langle k, S_\tau(t) \rangle}] = \left(\int_{\Omega_\theta} \exp(\tau\psi_\theta(k)) M(d\theta) \right)^{\lfloor \frac{t}{\tau} \rfloor}.$$

Using the fact that $a^n - b^n = (a-b) \sum_{j=0}^{n-1} a^j b^{n-j-1}$ and $|\exp(\tau\psi(k))| \leq 1$, we see that

$$\begin{aligned} |\exp(t\psi(k)) - E[e^{-i\langle k, S_\tau(t) \rangle}]| &= |\exp(t\psi(k)) - \exp(\tau\psi(k))^{\lfloor \frac{t}{\tau} \rfloor}| \\ &\quad + \exp(\tau\psi(k))^{\lfloor \frac{t}{\tau} \rfloor} - E[e^{-i\langle k, S_\tau(t) \rangle}] \\ &\leq |\exp(t\psi(k)) - \exp(\tau \lfloor \frac{t}{\tau} \rfloor \psi(k))| \\ &\quad + \lfloor \frac{t}{\tau} \rfloor |\exp(\tau\psi(k)) - \int_{\Omega_\theta} \exp(\tau\psi_\theta(k)) M(d\theta)|. \end{aligned} \tag{4}$$

Using Taylor expansion the last term may be bounded as

$$\begin{aligned} \lfloor \frac{t}{\tau} \rfloor |\exp(\tau\psi(k)) - \int_{\Omega_\theta} \exp(\tau\psi_\theta(k)) M(d\theta)| \\ &= \lfloor \frac{t}{\tau} \rfloor \left| \sum_{j=2}^{\infty} \frac{(\tau\psi(k))^j}{j!} - \int_{\Omega_\theta} \sum_{j=2}^{\infty} \frac{(\tau\psi_\theta(k))^j}{j!} M(d\theta) \right| \\ &\leq \tau^2 \lfloor \frac{t}{\tau} \rfloor |\psi(k)|^2 \exp(\tau|\psi(k)|) \\ &\quad + \tau^2 \lfloor \frac{t}{\tau} \rfloor \int_{\Omega_\theta} |\psi_\theta(k)|^2 \exp(\tau|\psi_\theta(k)|) M(d\theta). \end{aligned} \tag{5}$$

It follows from our assumption on ψ_θ that the integral is bounded for fixed $k \in \mathbb{R}^d$ and τ small enough and hence (5) converges to zero uniformly in t on compact sets. Since the first term in (4) converges to zero uniformly on compacta as well, the proof is complete.

Example 1. Let $X(t)$ be an operator stable process. Using the Jurek coordinate system [10] the log-characteristic function can be written as

$$\psi(k) = \int_{\|\theta\|=1} \int_0^\infty \left(e^{-i\langle k, r^E \theta \rangle} - 1 + \frac{i\langle k, r^E \theta \rangle}{1 + \|r^E \theta\|^2} \right) \frac{c}{r^2} dr M(d\theta)$$

for some scaling matrix E with eigenvalues whose real part is larger than $1/2$. By Theorem 1 this process can be approximated by generating steps via randomly (according

to M) choosing a direction Θ and generating a random variable with log-characteristic function

$$\tau\psi_{\Theta}(k) = \tau \int_0^{\infty} \left(e^{-i\langle k, r^E \Theta \rangle} - 1 + \frac{i\langle k, r^E \Theta \rangle}{1 + \|r^E \Theta\|^2} \right) \frac{c}{r^2} dr.$$

Example 2. Let $X(t)$ be a tempered operator stable process with uniform scaling; i.e. its log-characteristic function can be written (see [2]) as

$$\begin{aligned} \psi(k) &= \int_{\|\theta\|=1} \int_0^{\infty} \left(e^{-i\langle k, \theta \rangle r} - 1 + i\langle k, \theta \rangle r \right) \frac{e^{-r\rho(\theta)}}{r^{\alpha+1}} dr M(d\theta) \\ &= c \int_{\|\theta\|=1} (\langle ik, \theta \rangle + \rho(\theta))^{\alpha} - \rho(\theta)^{\alpha} - \alpha \langle ik, \theta \rangle \rho(\theta)^{\alpha-1} M(d\theta) \end{aligned}$$

for some $1 < \alpha < 2$, $c > 0$ and exponential taper ρ ; i.e. a bounded measurable function $\rho : S^{d-1} \rightarrow \mathbb{R}^+$, where $S^{d-1} = \{\theta \in \mathbb{R}^d : \|\theta\| = 1\}$. By Theorem 1 this process can be approximated by generating steps via randomly choosing a direction Θ (according to M) and generating a one-dimensional exponentially tempered stable random variable Y_{Θ} with tempering coefficient $\rho(\Theta)$ and time scale τ (see [1] for an efficient algorithm to generate tempered stable variables with given time scale and tempering coefficient), and letting $X_{\Theta} = \Theta Y_{\Theta}$.

3. Rate of convergence for tempered stables

In the case of most tempered stable processes we can go further and actually provide a rate at which the densities of the processes converge. We are going to show that the characteristic functions converge uniformly at a rate of $o(\log^2 n/n)$ which translates into L^2 multiplier convergence of the densities or uniform convergence if an arbitrarily small normal random variable is added to the process.

Definition 2. Let $\rho(\theta) \geq 0$ be a bounded measurable function on S^{d-1} , $1 < \alpha < 2$ and $a > 0$. Let

$$A_{\theta} : k \mapsto a (\langle ik, \theta \rangle + \rho(\theta))^{\alpha} - a\rho(\theta)^{\alpha} - \alpha a \rho(\theta)^{\alpha-1} i\langle k, \theta \rangle$$

be the *tempered fractional derivative symbol in the direction θ* . If $\rho(\theta) = 0$, then we call $A_{\theta} : k \mapsto a \langle ik, \theta \rangle^{\alpha}$ the *fractional derivative symbol in the direction of θ* .

We call ρ the *taper* and the extended real valued function

$$F_C : \theta \mapsto \lim_{\epsilon \rightarrow 0^+} \left(\frac{\text{esssup}_{\|\theta\|=1} \rho(\theta) + \epsilon}{\rho(\theta) + \epsilon} \right)^{2-\alpha}$$

the *normalised fractional content of the taper at θ* .

For a probability measure M on S^{d-1} , define the (*tempered*) *fractional derivative symbol* to be

$$A : k \mapsto \int_{\|\theta\|=1} A_\theta(k) M(d\theta).$$

We say that a tempered fractional derivative symbol is *full* if

$$\lambda_M = \min_{\|\eta\|=1} \int_{\|\theta\|=1} \langle \eta, \theta \rangle^2 M(d\theta) > 0.$$

It is easy to show that λ_M is the smallest eigenvalue of the co-variance matrix of M viewed as a measure on \mathbb{R}^d and is zero if and only if M is supported on a subspace.

Our main theorem is the following:

Theorem 2. *Let A be a full, tempered fractional derivative symbol and assume $F_C \in L^2(S^{d-1}, M(d\theta))$. Then there exists $C \geq 0$ such that*

$$\left| \left(\int_{\|\theta\|=1} e^{\frac{t}{n} A_\theta(k)} M(d\theta) \right)^n - e^{tA(k)} \right| \leq C \frac{1 + \log^2 n}{n} \quad (6)$$

for all $k \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $t \geq 0$.

Proof. See Section 4 below.

Let $\psi \in L^\infty(\mathbb{R}^d)$ and let $S(\mathbb{R}^d)$ denote the space of Schwartz functions. We call ψ an *L^p -Fourier multiplier*, $1 \leq p < \infty$, if the map

$$S(\mathbb{R}^d) \ni f \mapsto T_\psi f := \mathcal{F}^{-1}(\psi \mathcal{F}(f))$$

extends to a bounded linear operator on $L^p(\mathbb{R}^d)$, where the Fourier transform of f is denoted by $\mathcal{F}(f)(k) = \int_{\mathbb{R}^d} e^{-ikx} f(x) dx$. It is well-known that for a bounded Borel measure μ on \mathbb{R}^d , its Fourier transform $\hat{\mu}(k) = \int_{\mathbb{R}^d} e^{-ikx} \mu(dx)$ is an L^p -Fourier-multiplier and

$$T_{\hat{\mu}} f = \mu * f.$$

The Fourier multiplier p -norm of μ is defined as

$$\|\mu\|_{\mathcal{M}_p(\mathbb{R}^d)} := \sup_{\|f\|_{L^p(\mathbb{R}^d)}=1} \|\mu * f\|_{L^p(\mathbb{R}^d)} = \|T_{\hat{\mu}}\|_{\mathcal{B}(L^p(\mathbb{R}^d))},$$

where $*$ denotes convolution and $\|\cdot\|_{\mathcal{B}(L^p(\mathbb{R}^d))}$ denotes the operator norm on $L^p(\mathbb{R}^d)$.

Let μ_t and ν_t be probability measures with Fourier transforms

$$\hat{\mu}_t(k) = \int_{\|\theta\|=1} e^{tA_\theta(k)} M(d\theta)$$

and

$$\hat{\nu}_t(k) = e^{t \int_{\|\theta\|=1} A_\theta(k) M(d\theta)}$$

and let μ^{n*} denote the n -th convolution power of a measure μ .

Corollary 1. *Let A be a full, tempered fractional derivative symbol and assume $F_C \in L^2(S^{d-1})$. Then for all $\epsilon > 0$ there exists $C \geq 0$ such that*

$$\|\mu_t^{n*} - \nu_t\|_{\mathcal{M}_2(\mathbb{R}^d)} \leq C \frac{1 + \log^2 n}{n}$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

Proof. Since for a bounded Borel measure μ we have $\|\mu\|_{\mathcal{M}_2(\mathbb{R}^d)} = \sup_{k \in \mathbb{R}^d} |\hat{\mu}(k)|$, the statement follows from Theorem 2.

The next corollary translates L^2 multiplier convergence into uniform convergence in the presence of a small perturbation $\delta N(0, 1)$, where $N(0, 1)$ is the multivariate standard normal random variable.

Corollary 2. *Let X be a tempered stable process with characteristic function*

$$E[e^{-i\langle k, X(t) \rangle}] = \exp(tA(k))$$

where A is a full, tempered fractional derivative symbol with $F_C \in L^2(S^{d-1})$. For $j \in \mathbb{N}$, $\tau > 0$ and $\|\theta\| = 1$, let Θ_j and $Y_{j,\theta}(\tau)$ be random variables on the same probability space, all independent, with Θ_j distributed as $\Pr\{\Theta_j \in \Omega\} = M(\Omega)$ and the distribution of $Y_{j,\theta}$ satisfying

$$E[e^{-i\langle k, Y_{j,\theta}(\tau) \rangle}] = \exp(\tau A_\theta(k)).$$

Then, for all $\delta > 0$, there exists $C > 0$ such that the marginal densities of the approximate process $\delta N(0, 1) + \sum_{j=1}^{\lfloor t/\tau \rfloor} \Theta_j Y_{j,\theta}(\tau)$ converge uniformly in $x \in \mathbb{R}^d$ and $t \geq 0$ at a rate of $C\tau(1 + \log^2(1/\tau))$ to the marginal densities of $\delta N(0, 1) + X(t)$ as $\tau \rightarrow 0+$.

Proof. See Section 4.

The next theorem is important when it is used in conjunction with Corollary 1, as it allows the mixing of operators with different α 's or tempered and untempered ($\rho \equiv 0$) operators.

Theorem 3. *Let $A_j \in C(\mathbb{R}^d)$, $j = 1, \dots, m$ be sectorial; i.e., assume that there is $c > 0$ such that $\operatorname{Re}(A_j(k)) \leq -c|A_j(k)|$ for $k \in \mathbb{R}^d$ and all $j = 1, \dots, m$ and let μ_t and ν_t be probability measures with Fourier transforms $\hat{\mu}_t(k) = \sum \lambda_j e^{tA_j(k)}$ and $\hat{\nu}_t(k) = e^{t \sum \lambda_j A_j(k)}$. Then, for each collection of $0 < \lambda_j < 1$ with $\sum \lambda_j = 1$, there exists $C > 0$ such that*

$$\|\mu_{t/n}^{n*} - \nu_t\|_{\mathcal{M}_2(\mathbb{R}^d)} \leq C \frac{1 + \log^2 n}{n}$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

Proof. As in the proof of Corollary 1 we use the fact that

$$\|\mu_{t/n}^{n*} - \nu_t\|_{\mathcal{M}_2(\mathbb{R}^d)} = \left\| \left(\sum \lambda_j e^{\frac{t}{n} A_j} \right)^n - e^{t \sum \lambda_j A_j} \right\|_{L^\infty(\mathbb{R}^d)}. \quad (7)$$

Without loss of generality assume $0 < \lambda_1 \leq \lambda_j$, which implies that $\lambda_1 \leq 1/m$. We divide the proof into 2 cases.

Case 1. Assume that $k \in \mathbb{R}^d$ is such that

$$\sum \lambda_j \left| \frac{t}{n} A_j(k) \right| \leq 2m \log n / (c\lambda_1 n).$$

Then $\frac{t}{n} |A_j(k)| \leq 2m \log n / (\lambda_1^2 c n)$ and by the binomial formula

$$\begin{aligned} \left| \left(\sum \lambda_j e^{\frac{t}{n} A_j(k)} \right)^n - e^{t \sum \lambda_j A_j(k)} \right| &\leq n \left| \left(\sum \lambda_j e^{\frac{t}{n} A_j(k)} \right) - e^{\sum \lambda_j \frac{t}{n} A_j(k)} \right| \\ &\leq n \left(\sum \lambda_j \left| \frac{t}{n} A_j(k) \right|^2 e^{\frac{t}{n} |A_j(k)|} \right. \\ &\quad \left. + \left| \sum \lambda_j \frac{t}{n} A_j(k) \right|^2 e^{\sum \lambda_j \frac{t}{n} |A_j(k)|} \right) \\ &\leq nC (\log n/n)^2 = C \log^2 n/n. \end{aligned}$$

Case 2. Assume that $k \in \mathbb{R}^d$ is such that

$$\sum \lambda_j \left| \frac{t}{n} A_j(k) \right| \geq 2m \log n / (c\lambda_1 n).$$

Then there exists j_1 such that $\lambda_{j_1} \left| \frac{t}{n} A_{j_1}(k) \right| \geq 2 \log n / (c\lambda_1 n)$ and hence

$$\begin{aligned} \left| \sum \lambda_j e^{\frac{t}{n} A_j(k)} \right| &\leq \lambda_{j_1} e^{-c \left| \frac{t}{n} A_{j_1}(k) \right|} + (1 - \lambda_{j_1}) \\ &\leq \lambda_{j_1} e^{-2c \log n / c\lambda_1 n} + (1 - \lambda_{j_1}) \\ &= \lambda_{j_1} e^{-2 \log n / \lambda_1 n} + (1 - \lambda_{j_1}). \end{aligned}$$

It can be seen by differentiating with respect to x that if $0 \leq x \leq \ln 2$, then $\lambda e^{-x} + (1 - \lambda) \leq e^{-\lambda x/2}$ and hence, as $\lambda_1 \leq \lambda_{j_1}$,

$$\lambda_{j_1} e^{-2 \log n / \lambda_1 n} + (1 - \lambda_{j_1}) \leq e^{-\log n / n}.$$

Hence

$$\begin{aligned} \left| \left(\sum \lambda_j e^{\frac{t}{n} A_j(k)} \right)^n - e^{t \sum \lambda_j A_j(k)} \right| &\leq e^{-\log n} + e^{-2m \log n / \lambda_1} \\ &\leq C/n. \end{aligned}$$

Thus, by cases 1 and 2 above, there is C such that

$$\left\| \left(\sum \lambda_j e^{\frac{t}{n} A_j} \right)^n - e^{t \sum \lambda_j A_j} \right\|_{L^\infty(\mathbb{R}^d)} \leq C(1 + \log^2 n)/n, \quad n \in \mathbb{N}, \quad t \geq 0,$$

which finishes the proof in view of (7).

The next corollary allows us to approximate each $e^{\frac{t}{n} A_j}$ with its polar approximation.

Corollary 3. *Let $A_j = \int_{\|\theta\|=1} A_{j,\theta} M_j(d\theta)$, $j = 1, \dots, m$, be tempered fractional derivative operators each satisfying the conditions of Theorem 2 and let μ_t and ν_t be probability measures with Fourier transforms $\hat{\mu}_t(k) = \sum \lambda_j \int_{\|\theta\|=1} e^{t A_{j,\theta}(k)} M_j(d\theta)$ and $\hat{\nu}_t(k) = e^{t \sum \lambda_j A_j(k)}$. Then for each collection of $0 < \lambda_j < 1$ with $\sum \lambda_j = 1$ there exists $C > 0$ such that*

$$\|\mu_{t/n}^{n*} - \nu_t\|_{\mathcal{M}_2(\mathbb{R}^d)} \leq C \frac{1 + \log^2 n}{n}$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

Proof. Straightforward extension, combining the proofs of Theorems 2 and 3 using the cases for which

$$\sum \lambda_j \int_{\|\theta\|=1} \left| \frac{t}{n} A_{j,\theta}(k) \right| M_j(d\theta) \leq \frac{2m}{\lambda_1} \log n / \tilde{c}n$$

or not, with $\tilde{c} = c_S c_3$ where λ_1, c_S and c_3 are the smallest of the λ_i , the constants in Property (1) of Proposition 4.1 and the constants c from Proposition 4.2 respectively.

We finish this section with an example highlighting the intended usage of Corollaries 1, 2 and 3.

Example 3. Consider a process on \mathbb{R}^2 that has $\psi(k) = \frac{1}{2\pi} \int_0^{2\pi} A_\theta(k) d\theta$ as its log-characteristic function, where

$$A_\theta(k) = \begin{cases} \left(\langle ik, \vec{\theta} \rangle + \sin(\theta) \right)^{1.6} - \sin^{1.6}(\theta) - (1.6) \sin^{0.6}(\theta) \langle ik, \vec{\theta} \rangle & 0 \leq \theta \leq \pi \\ \langle ik, \vec{\theta} \rangle^{1.2} & \pi < \theta < 3\pi/2 \\ (2\pi - \theta) \langle ik, \vec{\theta} \rangle^{1.8} & 3\pi/2 < \theta < 2\pi, \end{cases}$$

and $\vec{\theta} = (\cos \theta, \sin \theta)$. In order to apply Corollary 3, let $\lambda_1 = 1/2, \lambda_2 = \lambda_3 = 1/4$, $M_1(dx) = \frac{1}{\pi} I_{[0, \pi]}(x) dx$, $M_2(dx) = \frac{2}{\pi} I_{[\pi, 3\pi/2]}(x) dx$, $M_3(dx) = \frac{8}{\pi^2} I_{[3\pi/2, 2\pi]}(x) (2\pi - x) dx$. So we can rewrite $\psi(k)$ as

$$\psi(k) = \lambda_1 \int_0^{2\pi} A_\theta(k) M_1(d\theta) + \lambda_2 \int_0^{2\pi} A_\theta(k) M_2(d\theta) + \lambda_3 \int_0^{2\pi} \frac{\pi}{4} \langle ik, \vec{\theta} \rangle^{1.8} M_3(d\theta).$$

By the developed theory the process can be faithfully approximated by generating increments with step-size τ , where each increment is generated by first generating a uniformly distributed random variable $\tilde{\Theta}$ over $[0, 2\pi]$ and letting $\Theta = \tilde{\Theta}$ for $\tilde{\Theta} < 3\pi/2$ and $\Theta = 2\pi - \sqrt{\pi^2 - \tilde{\Theta}\pi/2}$ otherwise and then generating a one-dimensional random variable X_Θ with characteristic function

$$E[\exp(-ikX_\Theta)] = \exp(\tau \tilde{A}_\Theta(k)),$$

where

$$\tilde{A}_\theta(k) = \begin{cases} (ik + \sin(\theta))^{1.6} - \sin^{1.6}(\theta) - (1.6) \sin^{0.6}(\theta) ik & 0 \leq \theta \leq \pi \\ (ik)^{1.2} & \pi < \theta < 3\pi/2 \\ \frac{\pi}{4} (ik)^{1.8} & 3\pi/2 < \theta < 2\pi. \end{cases}$$

The one-dimensional variables can be generated (or approximated) using the methods in [1, 6]. The increment is then given by $X_\Theta \vec{\Theta}$. In Figure 1 we plot a sample path over the time intervals $t < 1$, $t < 100$ and $t < 1000$, generated with $\tau = 1/1000$. In Figure 2 we plot the *log density* estimate at time $t = 1$ taken from 1,000,000 sample paths generated with $\tau = 1/64$. In Table 1 we show the absolute error decreasing between estimates for different τ . As the error from one simulation to the next with $\tau = 1/64$ is of the order of the error between simulations with decreased τ we stopped at $\tau = 1/64$.

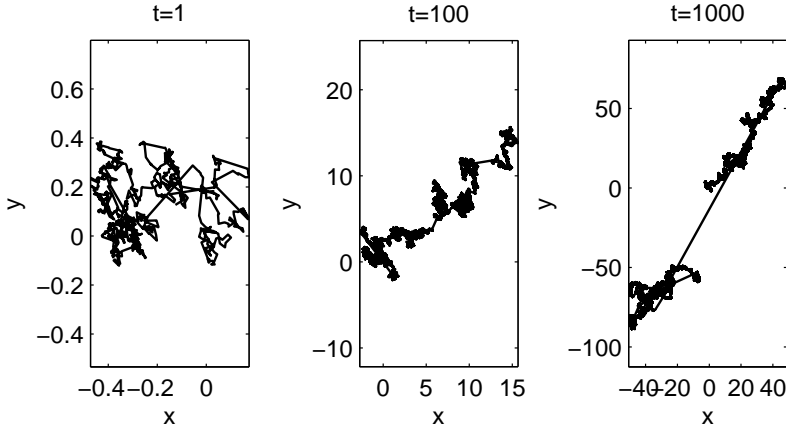


FIGURE 1: A simulated sample path of Example 3, generated with $\tau = 0.001$. Note that relatively large upwards jumps are present in the small and medium scale but virtually disappear in the larger scale on the right as the probability of jumps larger than x is less than $\exp(-x \sin(\theta))$ for $0 < \theta < \pi$.

τ	1	1/2	1/4	1/8	1/16	1/32	1/64
$\sup f_\tau - f_{1/64} $	0.107	0.072	0.045	0.020	0.008	0.004	0.003

TABLE 1: Absolute error in Monte-Carlo density estimates of 1,000,000 sample paths to $t = 1$ for different time steps. At $\tau = 1/64$ the error is between different realisations.

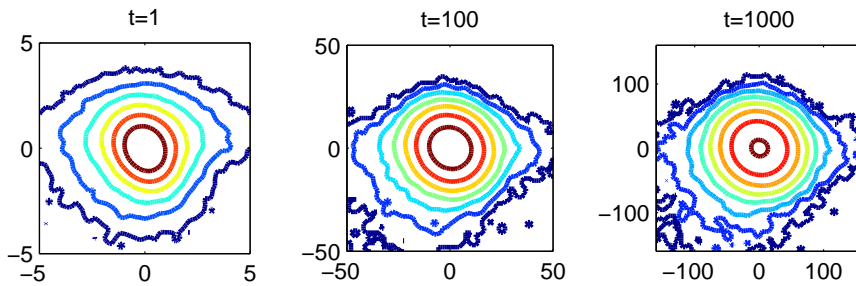


FIGURE 2: The estimated log densities of Example 3, generated with 1,000,000 sample paths and a small Gaussian smoothing. Level sets depict a decrease by 1.5 (in log space) starting innermost with -3 , -7.5 and -9 respectively.

4. Proof of Theorem 2 and Corollary 2

Proposition 4.1. *Let A be a tempered fractional derivative symbol. Let $u = \langle k, \theta \rangle$, where $k \in \mathbb{R}^d$ and $\theta \in S^{d-1}$. Then*

1. *There exists a constant $c_S > 0$, such that*

$$\operatorname{Re}(A_\theta(k)) \leq -c_S |A_\theta(k)|$$

for all k and θ ; i.e., A_θ and A are sectorial.

2. *There exist constants $c_L, c_U > 0$ such that*

$$c_L \min \left\{ \frac{u^2}{\rho(\theta)^{2-\alpha}}, |u|^\alpha \right\} \leq |A_\theta(k)| \leq c_U \min \left\{ \frac{u^2}{\rho(\theta)^{2-\alpha}}, |u|^\alpha \right\}$$

for all k and θ with $\rho(\theta) > 0$; if $\rho(\theta) = 0$, $c_L |u|^\alpha \leq |A_\theta(k)| \leq c_U |u|^\alpha$.

Proof. Note that without loss of generality we can set $\rho(\theta) = a = 1$ as the general case follows by replacing u (or equivalently k) with $u/\rho(\theta)$ and multiplying A_θ by $a\rho(\theta)^\alpha$. As

$$(iu + 1)^\alpha - 1 - \alpha iu = (iu)^\alpha + o(|u|^\alpha)$$

as $u \rightarrow \infty$ and by the Taylor expansion,

$$(iu + 1)^\alpha - 1 - \alpha iu = -\frac{\alpha(\alpha - 1)}{2} u^2 + o(u^2)$$

as $u \rightarrow 0$, the inequalities follow once we establish that $|\operatorname{Re}(A_\theta)|$ and $|\operatorname{Im}(A_\theta)|$ with $\rho(\theta) = 1$ are continuous, increasing functions of $|u|$.

First we show that $\operatorname{Re}((iu + 1)^\alpha - 1)$ is decreasing for $u > 0$; it clearly is continuous and differentiable. To that end let $\phi = \arctan u$. Then $|iu + 1| = \sec \phi$ and

$$f(\phi) = \operatorname{Re}((i \tan \phi + 1)^\alpha - 1) = \cos \alpha \phi \sec^\alpha \phi - 1. \quad (8)$$

Its derivative is given by

$$\begin{aligned} \frac{d}{d\phi} f(\phi) &= -\alpha \sin(\alpha \phi) \sec^\alpha(\phi) + \cos(\alpha \phi) \alpha \sec^{\alpha-1}(\phi) \sec(\phi) \tan(\phi) \\ &= -\alpha \sec^{\alpha+1}(\phi) (\sin(\alpha \phi) \cos(\phi) - \cos(\alpha \phi) \sin(\phi)) \\ &= -\alpha \sec^{\alpha+1}(\phi) \sin(\alpha \phi - \phi) < 0 \end{aligned} \quad (9)$$

for $\phi > 0$ and positive for $\phi < 0$. Hence f is decreasing for positive ϕ or u and increasing for negative u , and since $f(0) = 0$ we have that $|f|$ is increasing for increasing $|\phi|$ or $|u|$.

Similarly we show that $\text{Im}((iu + 1)^\alpha - \alpha iu)$ is increasing. Again let $\phi = \arctan u$ and

$$f(\phi) = \text{Im}((i \tan \phi + 1)^\alpha - \alpha i \tan \phi) = \sin \alpha \phi \sec^\alpha \phi - \alpha \tan \phi.$$

Then

$$\begin{aligned} \frac{d}{d\phi} f(\phi) &= \alpha \cos(\alpha \phi) \sec^\alpha(\phi) + \sin(\alpha \phi) \alpha \sec^{\alpha-1}(\phi) \sec(\phi) \tan(\phi) - \alpha \sec^2(\phi) \\ &= \alpha \sec^{\alpha+1}(\phi) (\cos(\alpha \phi) \cos(\phi) + \sin(\alpha \phi) \sin(\phi)) - \alpha \sec^2 \phi \\ &= \alpha \sec^{\alpha+1}(\phi) \cos(\alpha \phi - \phi) - \alpha \sec^2(\phi) \\ &= \sec^2(\phi) (\alpha \sec^{\alpha-1}(\phi) \cos((\alpha - 1)\phi) - \alpha). \end{aligned} \tag{10}$$

At $\phi = 0$ we have that $\frac{d}{d\phi} f(\phi) = 0$. The last factor in (10) is similar to (8) and its derivative is computed similarly to (9) and is given by

$$-\alpha(\alpha - 1) \sec^\alpha(\phi) \sin((\alpha - 2)\phi) > 0$$

for $\phi > 0$ and negative for $\phi < 0$. Hence $\frac{d}{d\phi} f(\phi) > 0$ for all $\phi \neq 0$ and since $f(0) = 0$ we have that $|f|$ is increasing for increasing $|u|$.

Lemma 4.1. *Let $F_C \in L^1(S^{d-1}, M(d\theta))$ and $\epsilon > 0$. Then there exists a constant $c > 0$ such that*

$$\int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) \geq \epsilon$$

implies that if $\rho_{\max} := \text{esssup}_{\|\theta\|=1} \rho^{2-\alpha}(\theta) > 0$,

$$\min \left\{ \|k\|^2 \frac{\|F_C\|_1}{\rho_{\max}}, \|k\|^\alpha \right\} > c\epsilon,$$

otherwise $\|k\|^\alpha > c\epsilon$.

Proof. Assume $\rho_{\max} > 0$. As $F_C \in L^1(S^{d-1})$, the set for which $\rho(\theta) = 0$ is a null

set. By Proposition 4.1,

$$\begin{aligned} \int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) &\leq c_U \int_{\|\theta\|=1} \min \left\{ \left| \frac{\langle k, \theta \rangle^2}{\rho(\theta)^{2-\alpha}} \right|, |\langle k, \theta \rangle|^\alpha \right\} M(d\theta) \\ &\leq c_U \min \left\{ \int_{\|\theta\|=1} \left| \frac{\langle k, \theta \rangle^2}{\rho(\theta)^{2-\alpha}} \right| M(d\theta), \int_{\|\theta\|=1} |\langle k, \theta \rangle|^\alpha M(d\theta) \right\} \\ &\leq c_U \min \left\{ \|k\|^2 \frac{\|F_C\|_1}{\rho_{\max}}, \|k\|^\alpha \right\}. \end{aligned}$$

Hence $\min \left\{ \|k\|^2 \frac{\|F_C\|_1}{\rho_{\max}}, \|k\|^\alpha \right\} \geq \epsilon/c_U$.

In case of $\rho_{\max} = 0$, $|A_\theta(k)| = a|\langle k, \theta \rangle|^\alpha$. Similarly we then obtain $\|k\|^\alpha \geq \epsilon/c_U$.

Lemma 4.2. *There exists $c > 0$ such that if $\rho_{\max} := \operatorname{esssup}_{\|\theta\|=1} \rho(\theta)^{2-\alpha} > 0$, then*

$$|A_\theta(k)| \geq c \min \left\{ \frac{\|k\|^2}{\rho_{\max}}, \|k\|^\alpha \right\} \frac{\langle k, \theta \rangle^2}{\|k\|^2},$$

otherwise $|A_\theta(k)| \geq c\|k\|^{\alpha-2} \langle k, \theta \rangle^2$.

Proof. If $\rho(\theta) = 0$, then $|A_\theta(k)| = a|\langle k, \theta \rangle|^\alpha \geq a\|k\|^\alpha \frac{\langle k, \theta \rangle^2}{\|k\|^2}$. If $\rho(\theta) > 0$, by Proposition 4.1 we have that

$$\begin{aligned} |A_\theta(k)| &\geq c_L \min \left\{ \left| \frac{\langle k, \theta \rangle^2}{\rho(\theta)^{2-\alpha}} \right|, |\langle k, \theta \rangle|^\alpha \right\} \\ &\geq c_L \min \left\{ \frac{|\langle k, \theta \rangle|^2}{\rho_{\max}}, \|k\|^\alpha \left| \left\langle \frac{k}{\|k\|}, \theta \right\rangle \right|^\alpha \right\} \\ &\geq c_L \min \left\{ \frac{\|k\|^2}{\rho_{\max}} \left\langle \frac{k}{\|k\|}, \theta \right\rangle^2, \|k\|^\alpha \left\langle \frac{k}{\|k\|}, \theta \right\rangle^2 \right\} \\ &= c_L \min \left\{ \frac{\|k\|^2}{\rho_{\max}}, \|k\|^\alpha \right\} \frac{\langle k, \theta \rangle^2}{\|k\|^2}. \end{aligned}$$

Lemma 4.3. *Let M be a probability measure on $[0, 1]$. Then for $\mu \geq 0$,*

$$\int_0^1 e^{-\mu u} M(du) \leq e^{-\frac{\min(\mu, 1)}{2}} \int_0^1 u M(du).$$

Proof. Clearly, for $u \in [0, 1]$, $e^{-\mu u} \leq 1 - (1 - e^{-\mu})u$. Hence

$$\int_0^1 e^{-\mu u} M(du) \leq 1 - (1 - e^{-\mu}) \int_0^1 u M(du) \leq e^{-(1-e^{-\mu})} \int_0^1 u M(du).$$

The assertion follows from the fact that for $0 \leq \mu \leq 1$, $1 - e^{-\mu} \geq \mu/2$ and for $\mu > 1$, $1 - e^{-\mu} \geq 1/2$.

Proposition 4.2. *Let A be a full, tempered fractional derivative symbol. Assume $F_C \in L^1(S^{d-1}, M(d\theta))$. Then there exist $0 < c \leq 1$ and $d > 0$ such that for all $\epsilon, t > 0$ and $n \in \mathbb{N}$,*

$$\int_{\|\theta\|=1} \frac{t}{n} |A_\theta(k)| M(d\theta) \geq \epsilon$$

implies that

$$\int_{\|\theta\|=1} e^{-\frac{t}{n} |A_\theta(k)|} M(d\theta) \leq e^{-\min\{c\epsilon, d\}}.$$

Proof. Combining Lemma 4.1 and Lemma 4.2, there exists $c_L, c_U > 0$ such that

$$\begin{aligned} |A_\theta(k)| &\geq c_L \min \left\{ \frac{\|k\|^2}{\rho_{\max}}, \|k\|^\alpha \right\} \frac{\langle k, \theta \rangle^2}{\|k\|^2} \\ &\geq \frac{n}{t} \frac{\epsilon c_L}{c_U} \min \left\{ \frac{1}{\|F_C\|_1}, 1 \right\} \frac{\langle k, \theta \rangle^2}{\|k\|^2} \\ &= \frac{n}{t} \frac{\epsilon c_L}{c_U \|F_C\|_1} \frac{\langle k, \theta \rangle^2}{\|k\|^2}. \end{aligned} \quad (11)$$

Define a measure M_k on $[0, 1]$ via

$$M_k(\Omega) = M(\{\theta : \langle k, \theta \rangle^2 / \|k\|^2 \in \Omega\})$$

for each measurable $\Omega \subset [0, 1]$. Then by Lemma 4.3 and inequality (11) there exists $0 < c \leq 1$ such that,

$$\begin{aligned} \int_{\|\theta\|=1} e^{-\frac{t}{n} |A_\theta(k)|} M(d\theta) &\leq \int_{\|\theta\|=1} e^{-\frac{t}{n} \frac{n}{t} \frac{\epsilon c_L}{c_U \|F_C\|_1} \frac{\langle k, \theta \rangle^2}{\|k\|^2}} M(d\theta) \\ &\leq \int_0^1 e^{-\frac{\epsilon c_L}{c_U \|F_C\|_1} u} M_k(du) \\ &\leq e^{-\frac{\min\{\frac{\epsilon c_L}{c_U \|F_C\|_1}, 1\}}{2}} \int_0^1 u M_k(du) \leq e^{-\frac{\min\{\frac{\epsilon c_L}{c_U \|F_C\|_1}, 1\}}{2}} \lambda_M \\ &\leq e^{-\min\{c\epsilon, d\}}. \end{aligned}$$

Lemma 4.4. *Let A be a full, tempered fractional derivative symbol. Then there exists a constant $c > 0$ such that for all $\epsilon > 0$,*

$$\int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) \leq \epsilon$$

implies that

$$|A_\theta(k)| < c\epsilon F_C(\theta).$$

Proof. By Lemma 4.2, if $\rho_{\max} := \text{esssup}_{\|\theta\|=1} \rho(\theta)^{2-\alpha} > 0$, then

$$|A_\theta(k)| \geq c_L \min \left\{ \frac{\|k\|^2}{\rho_{\max}}, \|k\|^\alpha \right\} \frac{\langle k, \theta \rangle^2}{\|k\|^2}$$

and hence

$$\epsilon \geq \int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) \geq c_L \min \left\{ \frac{\|k\|^2}{\rho_{\max}}, \|k\|^\alpha \right\} \lambda_M.$$

Therefore, either $\|k\|^2 \leq \epsilon \rho_{\max} / \lambda_M c_L$ or $\|k\|^\alpha \leq \epsilon / \lambda_M c_L$. If $\rho_{\max} = 0$ we clearly also have $\|k\|^\alpha \leq \epsilon / \lambda_M c_L$. In case of $\|k\|^2 \leq \epsilon \rho_{\max} / \lambda_M c_L$, this implies that

$$|A_\theta(k)| \leq c_U \min \left\{ \left| \frac{\langle k, \theta \rangle^2}{\rho(\theta)^{2-\alpha}} \right|, |\langle k, \theta \rangle|^\alpha \right\} \leq \frac{c_U}{\lambda_M c_L} F_C(\theta) \epsilon;$$

in case of $\|k\|^\alpha \leq \epsilon / \lambda_M c_L$, this implies that

$$|A_\theta(k)| \leq c_U \min \left\{ \left| \frac{\langle k, \theta \rangle^2}{\rho(\theta)^{2-\alpha}} \right|, |\langle k, \theta \rangle|^\alpha \right\} \leq \frac{c_U}{\lambda_M c_L} \epsilon.$$

As $F_C(\theta) \geq 1$, the lemma is proven.

Proposition 4.3. *Let A be a full, tempered fractional derivative symbol and assume that $F_C \in L^2(S^{d-1}, M(d\theta))$. Then there exist a constant $c > 0$ such that for all $0 < \epsilon \leq 1$ and all $n, t > 0$,*

$$\frac{t}{n} \int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) \leq \epsilon$$

implies that

$$\left| \int_{\|\theta\|=1} e^{\frac{t}{n} A_\theta(k)} M(d\theta) - e^{\frac{t}{n} A(k)} \right| \leq c\epsilon^2.$$

Proof. Note that by Lemma 4.4 there exists a constant $c = c_U / c_L \lambda_M$ such that

$$\frac{t}{n} |A_\theta(k)| \leq c\epsilon F_C(\theta).$$

Then, using the fact that $A(k) = \int_{\|\theta\|=1} A_\theta(k) M(d\theta)$ and that M is a probability

measure,

$$\begin{aligned}
& \left| e^{\frac{t}{n}A(k)} - \int_{\|\theta\|=1} e^{\frac{t}{n}A_\theta(k)} M(d\theta) \right| = \\
& = \left| \int_{\|\theta\|=1} \int_0^1 \left(\frac{t}{n}A(k) - \frac{t}{n}A_\theta(k) \right) e^{(s\frac{t}{n}A(k) + (1-s)\frac{t}{n}A_\theta(k))} ds M(d\theta) \right| \\
& = \left| \int_{\|\theta\|=1} s \left(\frac{t}{n}A(k) - \frac{t}{n}A_\theta(k) \right) e^{\frac{t}{n}(sA(k) + (1-s)A_\theta(k))} \right|_{s=0}^{s=1} \\
& \quad - \int_0^1 s \left(\frac{t}{n}A(k) - \frac{t}{n}A_\theta(k) \right)^2 e^{(s\frac{t}{n}A(k) + (1-s)\frac{t}{n}A_\theta(k))} ds M(d\theta) \right| \quad (12) \\
& = \left| \int_{\|\theta\|=1} \int_0^1 s \left(\frac{t}{n}A(k) - \frac{t}{n}A_\theta(k) \right)^2 e^{(s\frac{t}{n}A(k) + (1-s)\frac{t}{n}A_\theta(k))} ds M(d\theta) \right| \\
& \leq \frac{1}{2} \int_{\|\theta\|=1} \left| \frac{t}{n}A(k) - \frac{t}{n}A_\theta(k) \right|^2 M(d\theta) \\
& \leq \frac{1}{2} \int_{\|\theta\|=1} \left| \frac{t}{n}A_\theta(k) \right|^2 M(d\theta) + \frac{3}{2}\epsilon^2 \\
& \leq \frac{c^2\epsilon^2}{2} \int_{\|\theta\|=1} |F_C(\theta)|^2 M(d\theta) + \frac{3}{2}\epsilon^2.
\end{aligned}$$

Proof of Theorem 2. We divide the proof into two cases. Let first

$$\frac{t}{n} \int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) > \log n / \tilde{c}n,$$

where $\tilde{c} = c_S c_3$. Here c_S denotes the constant in Property (1) of Proposition 4.1 and $c_3 \leq 1$ is the constant c from Proposition 4.2. Then, by Property (1) of Proposition 4.1,

$$\left| e^{t \int_{\|\theta\|=1} A_\theta(k) M(d\theta)} \right| \leq e^{-nc_S \log n / \tilde{c}n} = 1/n^{1/c_3}.$$

Furthermore, by Proposition 4.2, we also have that

$$\left| \left(\int_{\|\theta\|=1} e^{\frac{t}{n}A_\theta(k)} M(d\theta) \right)^n \right| \leq e^{-c_S n \min\{c_3 \log n / \tilde{c}n, d\}}.$$

Hence, there is $C > 0$ such that

$$\left| \left(\int_{\|\theta\|=1} e^{\frac{t}{n}A(k)} \right)^n - e^{tA(k)} \right| \leq C/n.$$

In case

$$\frac{t}{n} \int_{\|\theta\|=1} |A_\theta(k)| M(d\theta) \leq \log n / \tilde{c}n,$$

first note that for $|a|, |b| \leq 1$,

$$|a^n - b^n| = |a - b| \sum_{j=0}^{n-1} |a^j b^{n-1-j}| \leq n|a - b|.$$

By Proposition 4.3 there exist $c > 0$ such that

$$\begin{aligned} \left| \left(\int_{\|\theta\|=1} e^{\frac{t}{n} A_\theta(k)} M(d\theta) \right)^n - \left(e^{\frac{t}{n} A(k)} \right)^n \right| &\leq n \left| \int_{\|\theta\|=1} e^{\frac{t}{n} A_\theta(k)} M(d\theta) - e^{\frac{t}{n} A(k)} \right| \\ &\leq nc(\log n / \tilde{c}n)^2 = C \frac{\log^2 n}{n}. \end{aligned} \quad (13)$$

Hence, combining the two cases, there exists a $C > 0$ such that (6) holds.

Proof of Corollary 2. Uniform convergence of the densities follows from the L^1 convergence of the characteristic function since $\|f\|_\infty \leq \frac{1}{(2\pi)^d} \|\hat{f}\|_1$, where $\hat{f} : k \mapsto \int e^{i\langle k, x \rangle} f(x) dx$. Hence we need to estimate

$$\begin{aligned} &\left\| e^{-\frac{\delta}{2} \|\cdot\|^2} \left(e^{tA(\cdot)} - \left(\int_{\|\theta\|=1} e^{\tau A_\theta(\cdot)} M(d\theta) \right)^{\lfloor \frac{t}{\tau} \rfloor} \right) \right\|_{L^1(\mathbb{R}^d)} \\ &\leq \left\| e^{-\frac{\delta}{2} \|\cdot\|^2} \left(e^{tA(\cdot)} - e^{\tau \lfloor \frac{t}{\tau} \rfloor A(\cdot)} \right) \right\|_{L^1(\mathbb{R}^d)} \\ &\quad + \left\| e^{-\frac{\delta}{2} \|\cdot\|^2} \left(e^{\tau \lfloor \frac{t}{\tau} \rfloor A(\cdot)} - \left(\int_{\|\theta\|=1} e^{\tau A_\theta(\cdot)} M(d\theta) \right)^{\lfloor \frac{t}{\tau} \rfloor} \right) \right\|_{L^1(\mathbb{R}^d)} \\ &= I_1 + I_2. \end{aligned}$$

Using that $\operatorname{Re} A(k) \leq 0$, we obtain

$$I_1 \leq \left\| e^{-\frac{\delta}{2} \|\cdot\|^2} \left| 1 - e^{\tau A(\cdot)} \right| \right\|_{L^1(\mathbb{R}^d)}.$$

Comparing real and imaginary parts we easily see that there exists $c > 0$ such that

$$\left| 1 - e^{\tau A(k)} \right| \leq c\tau (1 + |A(k)|)$$

and hence there exists C such that $I_1 \leq C\tau$.

In order to estimate I_2 , note that by Proposition 4.1 there exists c_U such that

$$\max_{\|\theta\|=1} |A_\theta(k)| \leq c_U \|k\|^\alpha$$

for all $k \in \mathbb{R}^d$. We divide the estimate into two parts.

Firstly, consider $t \leq 1$. Then using the same technique as in (12) and (13) we see that

$$\begin{aligned} \left| e^{\tau \lfloor \frac{t}{\tau} \rfloor A(k)} - \left(\int_{\|\theta\|=1} e^{\tau A_\theta(k)} M(d\theta) \right)^{\lfloor \frac{t}{\tau} \rfloor} \right| &\leq \frac{1}{2} \lfloor \frac{t}{\tau} \rfloor \tau^2 \int_{\|\theta\|=1} |A_\theta(k) - A(k)|^2 M(d\theta) \\ &\leq C\tau \|k\|^{2\alpha}, \end{aligned}$$

where the last inequality follows from Proposition 4.1. Hence, for $t \leq 1$,

$$I_2 \leq C\tau \left\| \|\cdot\|^{2\alpha} e^{-\frac{\delta}{2}(\|\cdot\|^2)} \right\|_{L^1(\mathbb{R}^d)}.$$

For $t \geq 1$, apply Theorem 2 with $n = \lfloor \frac{t}{\tau} \rfloor$ to obtain

$$\begin{aligned} \left| e^{\tau \lfloor \frac{t}{\tau} \rfloor A(k)} - \left(\int_{\|\theta\|=1} e^{\tau A_\theta(k)} M(d\theta) \right)^{\lfloor \frac{t}{\tau} \rfloor} \right| &\leq C_1 \frac{1 + \log^2 \lfloor \frac{t}{\tau} \rfloor}{\lfloor \frac{t}{\tau} \rfloor} \\ &\leq C\tau(1 + \log^2(1/\tau)) \end{aligned}$$

and hence in this case

$$I_2 \leq C\tau(1 + \log^2(1/\tau)) \left\| e^{-\frac{\delta}{2}\|\cdot\|^2} \right\|_{L^1(\mathbb{R}^d)}.$$

Thus, the marginal densities converge independently of t at the prescribed rate.

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