Tempered stable Lévy motion and transient super-diffusion

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Abstract

The space-fractional diffusion equation models anomalous super-diffusion. Its solutions are transition densities of a stable Lévy motion, representing the accumulation of power-law jumps. Tempered stable Lévy motion uses exponential tempering to cool these jumps. A tempered fractional diffusion equation governs the transition densities, which progress from super-diffusive early time to diffusive late-time behavior. This note provides finite difference and particle tracking methods for solving the tempered fractional diffusion equation with drift. A temporal and spatial 2nd-order Crank-Nicolson method is developed, based on a finite difference formula for tempered fractional derivatives. A new exponential rejection method for simulating tempered Lévy stables is presented to facilitate particle tracking codes.

1 Introduction

The diffusion equation $\partial_t p = c\partial_x^2 p$ governs the transition densities of a Brownian motion $B(t)$, and its solutions spread at the rate $t^{1/2}$ for all time. The space-fractional diffusion equation $\partial_t p = c\partial_x^\alpha p$ for $0 < \alpha < 2$ governs the transition densities of a totally skewed α-stable Lévy motion $S(t)$, and its solutions spread at the super-diffusive rate

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$S(ct) \sim c^{1/\alpha}S(t)$ for any time scale c. The super-diffusive spreading is the result of large power-law jumps, with the probability of a jump longer than r falling off like $r^{-\alpha}$ [11,15,16]. The super-diffusive scaling of a stable Lévy motion cannot be expressed in terms of the second moment, which fails to exist due to the power-law tail of the probability density [6]. Instead one can use fractional moments [17] or quantiles.

The totally skewed α-stable Lévy motion $S(t)$ is the scaling limit of a random walk with power-law jumps: $c^{-1/\alpha}(X_1+\cdots+X_{[ct]}) \Rightarrow S(t)$ in distribution as the time scale $c \to \infty$, when the independent jumps all satisfy $P(X > r) = Ar^{-\alpha}$ for large $x > 0$. Here $0 < \alpha < 2$ so that the second moment is infinite, and the usual Gaussian central limit theorem does not apply. The random walk with power-law jumps is called a Lévy flight [23]. The order of the fractional derivative in the governing equation $\partial_t p = c\partial_x^\alpha p$ equals the power law index of the jumps. A more general jump variable with $P(X < -r) = qAr^{-\alpha}$ and $P(X > r) = (1-q)Ar^{-\alpha}$ leads to a governing equation $\partial_t p = cq\partial_{-x}^\alpha p + (1-q)c\partial_x^\alpha p$ with $0 \le q \le 1$. A continuous time random walk (CTRW) imposes a random waiting time T_n before the jumps X_n, and power law waiting times $P(T > t) = Bt^{-\beta}$ for $0 < \beta < 1$ lead to a space-time fractional governing equation $\partial_t^\beta p = c\partial_x^\alpha p$. See [13] for more details. Fractional diffusion equations are important in applications to physics [15,16], finance [21], and hydrology [22].

Truncated Lévy flights were proposed by Mantegna and Stanley [9,10] as a modification of the α-stable Lévy motion, since many deem the possibility of arbitrarily long jumps, and the mathematical fact of infinite moments, physically unrealistic. In that model, the largest jumps are simply discarded. Some other modifications to achieve finite second moments were proposed by Sokolov et al. [24], who add a higher order power law factor, and Chechkin et al. [4], who add a nonlinear friction term. Tempered Lévy motion takes a different approach, exponentially tempering the probability of large jumps, so that very large jumps are exceedingly unlikely, and all moments exist [19]. Exponential tempering offers technical advantages, since the tempered process remains an infinitely divisible Lévy process whose governing equation can be identified, and whose transition densities can be computed at any scale. Those transition densities solve a tempered fractional diffusion equation, quite similar to the fractional diffusion equation [2]. Like the truncated Lévy flights, they evolve from super-diffusive early time behavior, to diffusive late-time behavior.

This paper provides finite difference and particle tracking methods for solving the tempered fractional diffusion equation. Higher order methods for fractional diffusion equations are based on the Grünwald finite difference approximation for the fractional derivative [14,25]. This paper develops a second order Crank-Nicolson scheme, using a variant of the Grünwald finite difference formula for tempered fractional derivatives combined with a Richardson extrapolation.

Particle tracking solutions for fractional diffusion equations take a Langevin approach, simulating a Markov process whose generator is the adjoint of the fractional derivative operator [27]. For tempered anomalous diffusion, particle tracking can be accomplished using an exponential rejection method for simulating tempered stables. Since the tempered stable process is still an infinitely divisible Lévy process, increments can be simulated for any choice of time step size.

2 Tempered stable Lévy motion

The totally skewed α-stable Lévy motion $x = S(t)$ for $0 < \alpha < 2$, $\alpha \neq 1$ has a probability density $p(x,t)$ that solves a fractional diffusion equation:

$$\partial_t p(x,t) = c\partial_x^\alpha p(x,t). \tag{1}$$

Take Fourier transforms (denoted by $\hat{f}(k) = \int e^{-ikx} f(x)\,dx$)) to get $\frac{d}{dt}\hat{p}(k,t) = c(ik)^\alpha \hat{p}(k,t)$, recognizing that the fractional derivative ∂_x^α corresponds to multiplication by $(ik)^\alpha$ in Fourier space. The point-source solution $\hat{p}(k,t) = e^{tc(ik)^\alpha}$ inverts to a totally skewed α-stable density with no closed form, except in a few exceptional cases [15,20]. The constant $c > 0$ for $1 < \alpha < 2$, and $c < 0$ for $0 < \alpha < 1$. The useful scaling property $t^{1/\alpha} p(t^{1/\alpha} x, t) = p(x,1)$ is evident from the Fourier transform. The process has heavy tails with $P(S(t) > r) \approx Ar^{-\alpha}$, so that any moment $\langle S(t)^n \rangle = \int x^n P(x,t)\,dx$ of order $n > \alpha$ diverges [6]; hence the need for tempering in some circumstances.

The basic idea of exponential tapering is to modify the density function to the form $e^{-\lambda x} p(x,t)$ and re-normalize: Lemma 2.2.1 on p. 67 in Zolotarev [28] shows that the Fourier transform $\int e^{-iux} p(x,t)\,dx = e^{tc(iu)^\alpha}$ has an analytic extension to $\Im(u) < 0$. Setting $u = k - i\lambda$ shows that $\int e^{-ikx} e^{-\lambda x} p(x,t)\,dx = e^{tc(\lambda + ik)^\alpha}$. For the special case $0 < \alpha < 1$, a simpler argument with Laplace transforms appears in Rosiński [19]. Since $k = 0$ yields the total mass, the density $e^{-tc\lambda^\alpha} e^{-\lambda x} p(x,t)$ integrates to one. The Fourier transform is easily computed: $e^{tc[(\lambda+ik)^\alpha - \lambda^\alpha]}$. Exponential tapering changes the mean: $i\frac{d}{dk} e^{tc[(\lambda+ik)^\alpha - \lambda^\alpha]}|_{k=0} = -ct\alpha\lambda^{\alpha-1}$. Centering yields

$$\hat{p}_\lambda(k,t) = e^{tc[(\lambda+ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}]}.$$

Invert the Fourier transform to get the mean-zero tempered stable density:

$$p_\lambda(x,t) = e^{-\lambda x} e^{ct(\alpha-1)\lambda^\alpha} p(x - ct\alpha\lambda^{\alpha-1}, t). \tag{2}$$

Since $\frac{d}{dt}\hat{p}_\lambda(k,t) = c[(\lambda+ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}]\hat{p}_\lambda(k,t)$, Fourier inversion reveals the tempered fractional diffusion equation

$$\partial_t p_\lambda(x,t) = c\,\partial_x^{\alpha,\lambda} p_\lambda(x,t) \tag{3}$$

where $\partial_x^{\alpha,\lambda} f(x)$ is the inverse Fourier transform of $[(\lambda+ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$. Any easy calculation [2] shows that

$$\partial_x^{\alpha,\lambda} f(x) = e^{-\lambda x}\,\partial_x^\alpha[e^{\lambda x}\,f(x)] - \lambda^\alpha f(x) - \alpha\lambda^{\alpha-1}\partial_x f(x), \tag{4}$$

which modifies the notation in [2] so that the tempered fractional diffusion equation (3) preserves the center of mass.

Everything above carries through without modification for the case $\alpha = 2$. In this case we can explicitly compute $p(x,t) = (4\pi tc)^{-1/2}e^{-x^2/(4tc)}$. After substituting into (2), a little algebra shows that $p_\lambda(x,t) = p(x,t)$. Hence tempering has no effect in the Gaussian case, and a tempered diffusion is exactly the same as a classical diffusion.

The case $\alpha = 1$ requires a different form $\hat{p}(k,t) = e^{tc(ik)\ln(ik)}$ to solve (1) with $c > 0$. Now the scaling property is $tp(tx + tc\ln t, t) = p(x,1)$. Apply the exponential tempering, normalize and center as before to derive the Fourier representation

$$\hat{p}_\lambda(k,t) = e^{tc[(\lambda+ik)\ln(\lambda+ik)-\lambda\ln\lambda-ik(1+\ln\lambda)]}. \tag{5}$$

Note that Lemma 2.2.1 in Zolotarev [28] also applies in this case. Invert to get the mean-zero tempered stable density

$$p_\lambda(x,t) = e^{-\lambda x}e^{tc\lambda}p(x - tc(1 + \ln\lambda + \ln(tc)), t) \tag{6}$$

that solves the tempered fractional diffusion equation (3), where $\partial_x^{\alpha,\lambda} f(x)$ is the inverse Fourier transform of $[(\lambda + ik)\ln(\lambda + ik) - \lambda\ln\lambda - ik(1 + \ln\lambda)]\hat{f}(k)$ when $\alpha = 1$.

To summarize, a totally positively skewed stable density with index $0 < \alpha < 2$ can be tempered so that moments of all orders exist. The positively skewed tempered stable density is obtained by: (i) multiplying the stable density by $e^{-\lambda x}$; (ii) rescaling by a constant that makes the resulting function integrate to one; and (iii) shifting by a constant that makes the resulting density function have mean zero. Then we have the following result.

Proposition 1 *Let $S(t)$ be a totally skewed α-stable Lévy process for $0 < \alpha < 2$ with density $p(x,t)$ and Fourier transform*

$$E[e^{-ikS(t)}] = \hat{p}(k,t) = \begin{cases} \exp\left[ct(ik)^\alpha\right] & \alpha \neq 1 \\ \exp\left[ctik\ln(ik)\right] & \alpha = 1. \end{cases} \tag{7}$$

Then the tempered α-stable Lévy process $S_\lambda(t)$ has density

$$p_\lambda(x,t) = \begin{cases} e^{-\lambda x+(\alpha-1)ct\lambda^\alpha}p(x - ct\alpha\lambda^{\alpha-1}, t) & \alpha \neq 1 \\ e^{-\lambda x+ct\lambda}p(x - ct(1 + \ln\lambda), t) & \alpha = 1. \end{cases} \tag{8}$$

Next we extend Proposition 1 to the general case, to accommodate both positive and negative particle jumps. If $S(t)$ is positively skewed, then $-S(t)$ is negatively skewed, with density $p(-x,t)$. For $0 < \alpha < 2, \alpha \neq 1$ the density solves a fractional diffusion equation $\partial_t p = c\partial_{-x}^\alpha p$ where ∂_{-x}^α corresponds to multiplication by $(-ik)^\alpha$ in Fourier space. Tempering multiplies the density by $e^{\lambda x}$, normalizes to total mass one, and centers, as before. The resulting tempered stable density has Fourier transform $e^{tc[(\lambda-ik)^\alpha-\lambda^\alpha+ik\alpha\lambda^{\alpha-1}]}$ which just replaces ik by $-ik$ in the formula (5). It solves the tempered space-fractional diffusion equation $\partial_t p = c\,\partial_{-x}^{\alpha,\lambda} p$ where $\partial_{-x}^{\alpha,\lambda} f(x)$ is the inverse Fourier transform of $[(\lambda-ik)^\alpha-\lambda^\alpha+ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$. A similar sign change yields the negatively skewed tempered stable density and governing equation in the case $\alpha=1$. For the general case, generate two independent positively skewed tempered stable processes $S_\lambda^+(t)$ and $S_\lambda^-(t)$ with the same index α, and set

$$S_\lambda(t) = qS_\lambda^-(t) + (1-q)S_\lambda^+(t) \tag{9}$$

where $0 \le q \le 1$. The probability densities of this process solve the general tempered space-fractional diffusion equation

$$\partial_t p = cq\,\partial_{-x}^{\alpha,\lambda} p + c(1-q)\,\partial_x^{\alpha,\lambda} p. \tag{10}$$

Another interesting characterization of the tempered stable process can be stated in terms of its Lévy representation. In short, a Lévy process is the limiting case of a compound Poisson process whose jumps are described by the Lévy measure (e.g., see [12, Remark 3.1.18], and (38) below). The remainder of this section is devoted to showing that the tempered stable process can also be defined in terms of an exponentially tempered Lévy measure. In the process, we will also show that exponential tempering of the density function can also be used for any Lévy process with positive jumps.

Any infinitely divisible process $S(t)$ on $\mathbb{R}$ can be characterized by its Fourier transform

$$\mathbb{E}[e^{-ikS(t)}] = \exp\left[t\left(-aik-\sigma^2k^2+\int_{x\neq 0}\left(e^{-ikx}-1+\frac{ikx}{1+x^2}\right)\phi(dx)\right)\right] \tag{11}$$

where $a\in\mathbb{R}$, $\sigma\ge 0$ and the Lévy measure ϕ with $\int \frac{x^2}{1+x^2}\phi(dx)<\infty$, are uniquely determined [6,12]. We call $[a,\sigma,\phi]$ the Lévy representation.

Given an infinitely divisible process $S(t)$ with Lévy representation $[0,0,\phi]$, we define the exponentially tempered Lévy measure $\phi_\lambda(dx)=\exp(-\lambda|x|)\phi(dx)$. It is clear that $\int \frac{x^2}{1+x^2}\phi_\lambda(dx)<\infty$, so that there exists another infinitely divisible Lévy process $S_\lambda(t)$ with Lévy representation $[a,0,\phi_\lambda]$ for any $a\in\mathbb{R}$. Write $\phi=\phi_++\phi_-$ where $\phi_+(U)=\phi(U\cap\{x: x>0\})$ is the positive part of the Lévy measure. It is clear from (11) that we can write $S(t)=S^+(t)+S^-(t)$ where $S^+(t)$ is infinitely divisible with Lévy representation $[0,0,\phi_+]$, $S^-(t)$ is infinitely divisible with Lévy representation $[0,0,\phi_-]$, and $S^+(t), S^-(t)$ are independent. In order to develop a theory of tempered infinitely divisible laws, it suffices to consider just the positive case. The extension to the signed case is the same as for tempered stable laws. Hence it suffices to consider infinitely divisible laws whose Lévy measure is concentrated on the positive real line.

Theorem 2 *Let $S(t)$ be an infinitely divisible process with Lévy representation $[a,\sigma,\phi]$, where ϕ is supported on $\mathbb{R}^+$. Write the Fourier transform $\mathbb{E}[e^{-ikS(t)}]=\exp(t\Phi(k))$ and* *the probability distribution $P(U,t) = \int_U P(dx,t) = \mathbb{P}(S(t) \in U)$ for Borel measurable $U \subset \mathbb{R}$. Then Φ has an analytic extension to the lower half plane. Given $\lambda > 0$, define an infinitely divisible process $S_\lambda(t)$ with Lévy representation $[d, \sigma, \phi_\lambda]$ where $\phi_\lambda(dx) = e^{-\lambda x}\phi(dx)$. The mean $\mathbb{E}[S_\lambda(t)]$ exists for all $\lambda > 0$, and we can choose $d \in \mathbb{R}$ so that $\mathbb{E}[S_\lambda(t)] = 0$. The resulting tempered infinitely divisible Lévy process $S_\lambda(t)$ has probability distribution*

$$P_\lambda(dx,t) = e^{-\lambda x - \Phi(-i\lambda)t - i\lambda\Phi'(-i\lambda)t} P(dx + i\Phi'(-i\lambda)t, t). \tag{12}$$

Proof. First we show that

$$\Phi(k) = -aik - \sigma^2 k^2 + \int_0^\infty \left(e^{-ikx} - 1 + \frac{ikx}{1+x^2} \right) \phi(dx)$$

has an analytic extension to the lower half plane. For $\Im(z) < 0$, let

$$\Psi(z) = \int_0^\infty \frac{1+x^2}{x^2} \left(e^{-izx} - 1 + \frac{izx}{1+x^2} \right) \frac{x^2}{1+x^2} \phi(dx).$$

As ϕ is a Lévy measure, $\int_0^\infty \frac{x^2}{1+x^2} \phi(dx) < \infty$. The rest of the integrand satisfies

$$\begin{aligned} \left| \frac{1+x^2}{x^2} \left(e^{-izx} - 1 + \frac{izx}{1+x^2} \right) \right| &= \left| \frac{e^{-izx} - 1 + izx}{x^2} + e^{-izx} - 1 \right| \\ &< \begin{cases} |z^2| \exp(|z|x) + 2 & x < 1 \\ (2 + |z|x)/x^2 + 2 & x \geq 1. \end{cases} \end{aligned} \tag{13}$$

Hence Ψ is well defined, its complex derivative

$$\begin{aligned} \lim_{w \to 0} \frac{1}{w} \int_0^\infty \frac{1+x^2}{x^2} &\left(e^{-i(z+w)x} - 1 + \frac{i(z+w)x}{1+x^2} - \left(e^{-izx} - 1 + \frac{izx}{1+x^2} \right) \right) \frac{x^2}{1+x^2} \phi(dx) \\ &= \lim_{w \to 0} \frac{1}{w} \int_0^\infty \left(\frac{e^{-izx}\left(e^{-iwx} - 1\right) + iwx}{x^2} + e^{-izx}\left(e^{-iwx} - 1\right) \right) \frac{x^2}{1+x^2} \phi(dx) \\ &= -i \int_0^\infty \left(\frac{e^{-izx} - 1}{x} + xe^{-izx} \right) \frac{x^2}{1+x^2} \phi(dx) \end{aligned} \tag{14}$$

exists by the Dominated Convergence Theorem as

$$\begin{aligned} \left| \frac{1}{w} \frac{e^{-izx}\left(e^{-iwx} - 1\right) + iwx}{x^2} + e^{-izx} \frac{e^{-iwx} - 1}{w} \right| < &\left| \frac{1}{w} \frac{e^{-izx}\left(e^{-iwx} - 1\right) + iwx}{x^2} \right| \\ &+ \left| e^{-izx} \frac{e^{-iwx} - 1}{w} \right|. \end{aligned} \tag{15}$$

The second term is bounded by $|e^{-izx}| x e^{|w|x} < x e^{(\Im(z) + |w|)x} < M$ for $\Im(z) + |w| < 0$. For $x > 1$, almost the same bound applies to the first term. For $x < 1$, a Taylor expansion shows that

$$\left|\frac{e^{-izx}\left(e^{-iwx}-1\right)+iwx}{wx^2}\right| < M. \tag{16}$$

Note that

$$\begin{aligned}\Psi(k-i\lambda)-\Phi(k)-aik-\sigma^2k^2 &= \int_0^\infty e^{-ikx-\lambda x}-e^{-ikx}-\frac{\lambda x}{1+x^2}\,\phi(dx)\\ &= \int_0^\infty e^{-ikx}\left(e^{-\lambda x}-1\right)-\frac{\lambda x}{1+x^2}\,\phi(dx)\end{aligned} \tag{17}$$

tends to zero as $\lambda \to 0^+$ by another dominated convergence argument, similar to (15) without the $1/w$. Hence $\Psi(z)-aiz-\sigma^2z^2$ is the analytic extension of Φ. In other words,

$$\Phi(k-i\lambda) = -ai(k-i\lambda)-\sigma^2(k-i\lambda)^2+\int_0^\infty \left(e^{-(ik+\lambda)x}-1+\frac{(ik+\lambda)x}{1+x^2}\right)\phi(dx)$$

is the unique extension of Φ.

Now

$$\begin{aligned}&\int_0^\infty \left(e^{-ikx}-1+\frac{ikx}{1+x^2}\right)e^{-\lambda x}\,\phi(dx)\\ &= \Phi(k-i\lambda)-\Phi(-i\lambda)+aik+\sigma^2k^2-2ik\lambda\sigma^2+bik\end{aligned} \tag{18}$$

for $b=\int_0^\infty (e^{-\lambda x}-1)\frac{x}{1+x^2}\phi(dx)\in\mathbb{R}$. Let $H(dx,t)$ denote the probability distribution of the infinitely divisible process with Lévy representation $[a,\sigma,\phi_\lambda]$, and denote its Fourier transform by $\hat{H}(k,t)=\int e^{-ikx}H(dx,t)$. Then

$$\hat{H}(k,t) = \exp\left[t\left(\Phi(k-i\lambda)-\Phi(-i\lambda)-2ik\lambda\sigma^2+bik\right)\right].$$

As $\hat{H}$ is infinitely differentiable, by [1, Prop. 5.1.19] all moments are finite. The mean is given by $t\mu = i\frac{\partial}{\partial k}\hat{H}(0,t) = it\Phi'(-i\lambda)+2t\lambda\sigma^2-bt$. Setting $d=a-\mu$ yields an infinitely divisible process $S_\lambda(t)$ with Lévy representation $[d,\sigma,\phi_\lambda]$ satisfying $\mathbb{E}[S_\lambda(t)]=0$ for all $t>0$. Its probability distribution $P_\lambda(dx,t)$ has Fourier transform

$$\hat{P}_\lambda(k,t) = \exp\left[t\left(\Phi(k-i\lambda)-\Phi(-i\lambda)+iki\Phi'(-i\lambda)\right)\right]. \tag{19}$$

By [26, Thm. III.10], the moment sequence $\{\mu_n\}_0^\infty$ with $\mu_n = i^n\frac{\partial^n}{\partial k^n}\hat{p}_\lambda(0,t)$ is positive (in the sense of [26, Def. III.9b]). Let $q(s)=\hat{P}_\lambda(-is,t)$. Then $\mu_n=(-1)^n\frac{d^n}{ds^n}q(0)$ and hence by [26, Thm. VI.19.c], the integral

$$q(s)=\hat{P}_\lambda(-is,t)=\int_{-\infty}^\infty e^{-sx}\,P_\lambda(dx,t)$$

exists for some measure P_λ and all $\Re(s) > -\lambda$. Therefore

$$\hat{P}_\lambda(k - i\nu, t) = \int_{-\infty}^{\infty} e^{-ikx} e^{-\nu x} \, P_\lambda(dx, t)$$

for all $k \in \mathbb{R}$ and $\nu > -\lambda$. As by (19)

$$\lim_{\nu \to -\lambda^+} \hat{P}_\lambda(k - i\nu, t) = \exp\left[t\left(\Phi(k) - \Phi(-i\lambda) + (ik - \lambda)i\Phi'(-i\lambda)\right)\right]$$

we obtain, using the shift formula for the Fourier transform, that the inverse Fourier transforms are the same as well:

$$e^{\lambda x} P_\lambda(dx, t) = e^{-\Phi(-i\lambda)t - i\lambda\Phi'(-i\lambda)t} P(dx + i\Phi'(-i\lambda)t, t),$$

which concludes the proof. $\square$

Remark: If $S(t)$ has a Lebesgue density $p(x,t)$, so that $P(dx,t) = p(x,t)dx$, then it follows immediately from Theorem 2 that the tempered process $S_\lambda(t)$ has density

$$p_\lambda(x,t) = e^{-\lambda x - \Phi(-i\lambda)t - i\lambda\Phi'(-i\lambda)t} p(x + i\Phi'(-i\lambda)t, t). \tag{20}$$

If we take $\Phi(k) = c(ik)^\alpha$ for $\alpha \neq 1$, or $\Phi(k) = c(ik)\log(ik)$ for $\alpha = 1$, then it follows by an easy computation that (8) holds. This shows that exponentially tempering the stable density is equivalent to exponentially tempering the corresponding Lévy measure. More generally, (20) shows that exponentially tempering the Lévy measure of an infinitely divisible law is equivalent to exponentially tempering the density.

3 Finite difference methods

In this section we develop a second order finite difference method to solve the tempered fractional advection dispersion equation with drift on a bounded interval $[x_L, x_R]$ with Dirichlet boundary conditions:

$$\partial_t u(x,t) = -v(x)\partial_x u(x,t) + c(x)\partial_x^{\alpha,\lambda} u(x,t) + q(x,t) \tag{21}$$

with $u(x,0) = u_0(x)$, $u(x_R,t) = B_R(t)$, and $u(x,t) = 0$ for all $x < x_L$ and $t > 0$. We assume $v(x) \geq 0$, $c(x) \geq 0$ and $1 < \alpha \leq 2$, to be consistent with the fractional advection dispersion equation for left-to-right flow [14]. Our approach is similar to [25]. First we develop an implicit Crank-Nicolson scheme that is second order in time, but fails to be second-order in space, due to the fact that the finite difference approximation of the fractional derivative is only first order. Then we apply Richardson extrapolation to obtain a method that is second order in both variables. We prove second-order consistency, and stability, and conclude with a numerical example.

Recall the fractional binomial formula

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = (1+z)^\alpha \tag{22}$$

which is valid for any $\alpha > 0$ and complex $|z| \leq 1$. Define

$$w_k = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = (-1)^k \binom{\alpha}{k}$$

and note that w_k can be computed recursively via $w_0 = 1, w_1 = -\alpha, w_{k+1} = \frac{k-\alpha}{k+1} w_k$. Our first result establishes a suitable finite difference approximation for the tempered fractional derivative. Recall that L^p denotes the class of functions f for which $\int |f(x)|^p \, dx$ exists, and the Sobolev space $W^{k,p}$ contains the L^p functions whose derivatives of order $j = 1, 2, \ldots, k$ are also in L^p.

Proposition 3 *Let $f \in W^{n+3,1}$ for some integer $n \geq 1$. Let $p \in \mathbb{R}, h > 0, \lambda \geq 0$ and $1 < \alpha \leq 2$. Define*

$$\Delta_{h,p}^{\alpha} f(x) := \frac{1}{h^\alpha} \sum_{j=0}^{\infty} \left(w_j e^{-(j-p)h\lambda} f(x-(j-p)h) \right) - e^{ph\lambda} \frac{\left(1-e^{-h\lambda}\right)^\alpha}{h^\alpha} f(x). \tag{23}$$

Then there exist constants a_j independent of h, f, x and λ such that

$$\begin{aligned} \Delta_{h,p}^{\alpha} f(x) = & \, \partial_x^{\alpha,\lambda} f(x) + \alpha \lambda^{\alpha-1} f'(x) \\ & + \sum_{j=1}^{n-1} a_j \left(e^{-\lambda x} \partial_x^{\alpha+j} \left[e^{\lambda x} f(x) \right] - \lambda^{\alpha+j} f(x) \right) h^j + O(h^n) \end{aligned} \tag{24}$$

uniformly in $x \in \mathbb{R}$.

Proof. The proof is similar to [25, Proposition 3.1]. Use the fractional binomial formula (22) with $z = -e^{-h(\lambda+ik)}$, along with the fact that $e^{-ika}\hat{f}(k)$ is the Fourier transform of $f(x-a)$, to see that (23) has Fourier transform $\phi_h(k)\hat{f}(k)$ with

$$\begin{aligned} \phi_h(k) = & \frac{1}{h^\alpha} \left(\sum_{j=0}^{\infty} w_j e^{-(j-p)h(\lambda+ik)} - e^{ph\lambda} \left(1-e^{-h\lambda}\right)^\alpha \right) \\ = & e^{ph(\lambda+ik)} \left(\frac{1-e^{-h(\lambda+ik)}}{h} \right)^\alpha - e^{ph\lambda} \left(\frac{1-e^{-h\lambda}}{h} \right)^\alpha \\ = & (\lambda+ik)^\alpha \omega_h(\lambda+ik) - \lambda^\alpha \omega_h(\lambda) \end{aligned} \tag{25}$$

where

$$\omega_h(z) = e^{phz} \left(\frac{1-e^{-hz}}{hz} \right)^\alpha .$$

A Taylor series expansion and some elementary estimates imply

$$\left| \omega_h(\lambda + ik) - \sum_{j=0}^{n-1} a_j (\lambda + ik)^j h^j \right| \le C h^n |\lambda + ik|^n e^{ph\lambda}$$

for some constants a_j and C independent of h. Hence

$$\phi_h(k)\hat{f}(k) = \sum_{j=0}^{n-1} a_j \left((\lambda + ik)^{\alpha+j} - \lambda^{\alpha+j} \right) h^j \hat{f}(k) + \hat{\psi}(k,h)$$

for some function ψ with

$$|\hat{\psi}(k,h)| \le C h^n (|\lambda + ik|^{\alpha+n} + \lambda^{\alpha+n}) e^{ph\lambda} |\hat{f}(k)|.$$

Since $f \in W^{n+3,1}$, a standard argument using the Riemann-Lebesgue Lemma implies that $|\hat{f}(k)| \le C(1 + |k|)^{-n-3}$. Hence $k \mapsto \psi(k,h) \in L^1(\mathbb{R})$ and $|\psi(x,h)| \le C|h|^n e^{ph\lambda}$ for all $x \in \mathbb{R}$, and then Fourier inversion yields (24) uniformly over $x \in \mathbb{R}$. □

Remark 4 *Note that the approximation* (24) *has terms involving* $\partial_x^{\alpha+j} f(x)$. *For these to be uniformly bounded,* f *has to be sufficiently regular, which is the case if* $f \in W^{n+3,1}$. *However, if for example* $f(x) = x^\beta$, *then* $\partial_x^{\alpha+j} f(x)$ *is not bounded near* $x = 0$ *if* $\alpha + j > \beta$, *and* (24) *will not hold. One should be able to rectify this by using the* starting quadrature weights *of Lubich [8], but we have not yet attempted this.*

Next we outline a Crank-Nicolson scheme for solving the tempered fractional diffusion equation (21) on $x \in [x_L, x_R]$ and $t \in [0,T]$. Define $t_n = n\Delta t$ to be the integration time $0 \le t_n \le T$ and $\Delta x = h > 0$ to be the spatial grid size with $x_i = x_L + i\Delta x$, $i = 0, \ldots, N_x$. Define $u_i^n = u(x_i, t_n)$, $c_i = c(x_i)$, $v_i = v(x_i)$, and $q_i^{n+1/2} = q(x_i, (t_{n+1} + t_n)/2)$. Let U_i^n define the numerical approximation to the exact solution u_i^n. In view of Proposition 3 we will write

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \left(-\frac{v_i + c_i \alpha \lambda^{\alpha-1}}{2} \delta_x + \frac{c_i}{2} \delta_x^{\alpha,\lambda} \right) \left(U_i^{n+1} + U_i^n \right) + q_i^{n+1/2}, \tag{26}$$

where $\delta_x U_i^n = (U_i^n - U_{i-1}^n)/\Delta x$ and

$$\delta_x^{\alpha,\lambda} U_i^n = \frac{1}{(\Delta x)^\alpha} \left(\sum_{k=0}^{i+1} w_k e^{-(k-1)\lambda\Delta x} U_{i-k+1}^n \right) - \frac{e^{\lambda \Delta x}}{(\Delta x)^\alpha} \left(1 - e^{-\lambda\Delta x} \right)^\alpha U_i^n. \tag{27}$$

Rearranging (26) for U_i^{n+1} in matrix form yields

$$(I - \Delta t A)\underline{U}^{n+1} = (I + \Delta t A)\underline{U}^n + \underline{Q}^{n+1/2} \Delta t, \tag{28}$$

where

$$\begin{aligned} \underline{U}^n &= [U_1^n, \ldots, U_{N_x-1}^n]^T, \\ \underline{Q}^{n+1/2} &= [q_1^{n+1/2}, q_2^{n+1/2}, \ldots, q_{N_x-1}^{n+1/2} + (\eta_{N_x-1} e^{\lambda\Delta x})(B_R^{n+1} + B_R^n)]^T, \end{aligned} \tag{29}$$

$\eta_i = \frac{c_i}{2(\Delta x)^\alpha}$ and I is the $(N_x - 1) \times (N_x - 1)$ identity matrix. The entries of the matrix A are collected from (26) to yield

$$A_{i,j} = \begin{cases} \eta_i e^{-\lambda(i-j)\Delta x} w_{i-j+1} & j \le i-2 \\ \eta_i e^{-\lambda \Delta x} w_2 + \frac{c_i \alpha \lambda^{\alpha-1} + v_i}{2\Delta x} & j = i-1 \\ \eta_i \left(w_1 - e^{\lambda \Delta x} \left(1 - e^{-\lambda \Delta x} \right)^\alpha \right) - \frac{c_i \alpha \lambda^{\alpha-1} + v_i}{2\Delta x} & j = i \\ \eta_i e^{\lambda \Delta x} & j = i+1 \\ 0 & j > i+1 \end{cases} \tag{30}$$

for $i, j = 1, \ldots, N_x - 1$. The matrix A is lower triangular plus entries on the first super-diagonal $j = i+1$. Note that the boundary condition $u(x_{N_x}, t_n) = B_R(t_n) = B_R^n$ is incorporated in the forcing term $\underline{Q}$, essentially to account for the fact that the super-diagonal term is cut off in the bottom row.

Theorem 5 *The Crank-Nicolson scheme* (28) *is consistent.*

Proof. The proof follows immediately from Proposition 3, along with the well-known consistency for the first order difference approximation $\delta_x U_i^n = (U_i^n - U_{i-1}^n)/\Delta x$. □

Proposition 6 *The fractional Crank-Nicolson scheme* (28) *is unconditionally stable.*

Proof. The proof is similar to [25, Proposition 3.3]. Note that (22) implies

$$\sum_{j=0}^{\infty} w_j e^{-(j-1)\lambda \Delta x} = e^{\lambda \Delta x} (1 - e^{-\lambda \Delta x})^\alpha .$$

It follows by a straightforward calculation that $\sum_{j=-\infty}^{i+1} A_{i,j} = 0$. Since the only negative entries of A are on the diagonal, this implies that $-A_{i,i} > \sum_{j=0, j \neq i}^{i+1} A_{i,j}$. Hence the matrix is diagonally dominant and using the Greshgorin theorem [7, pp. 135–136], all the eigenvalues of the matrix A in (30) have negative real parts. Then the spectral mapping theorem shows that the eigenvalues of $(I - \Delta t A)^{-1}(I + \Delta t A)$ have complex absolute value less than 1, so the system is unconditionally stable. □

Some standard arguments along with Proposition 3 show that the Crank-Nicolson scheme is $O(\Delta t^2 + \Delta x)$. We can improve the order of approximation by using Richardson extrapolation to obtain second order convergence; i.e., solve the system twice, once with grid size Δx and again with grid size $\Delta x/2$. Let $U_i = 2U_{2i, \Delta x/2} - U_{i, \Delta x}$ be the extrapolated solution on the coarse grid. Then, in view of Proposition 3, the extrapolated scheme is $O(\Delta t^2 + \Delta x^2)$.

Example 7 *Consider the tempered fractional diffusion equation with no drift*

$$\partial_t u(x,t) = c(x) \left(e^{-\lambda x} \frac{\partial^\alpha}{\partial x^\alpha} e^{\lambda x} u(x,t) - \lambda^\alpha u(x,t) - \alpha \lambda^{\alpha-1} \frac{\partial}{\partial x} u(x,t) \right) + q(x,t)$$

Table 1
Maximum error behavior for Crank-Nicolson (CN) and extrapolated (CNX) scheme showing improved second order convergence.

Δt	Δx	CN Max Error	CN Error rate	CNX Max Error	CNX Error rate
1/10	1/50	7.7738×10^{-5}	–	2.8514×10^{-6}	–
1/20	1/100	3.8353×10^{-5}	2.03	7.2120×10^{-7}	3.95
1/40	1/200	1.9055×10^{-5}	2.01	1.8157×10^{-7}	3.97
1/80	1/400	9.4976×10^{-6}	2.01	4.5555×10^{-8}	3.99

with $x \in [0, 1]$, $\alpha = 1.6$, $\lambda = 2$, *initial condition*

$$u(x, 0) = \frac{x^\beta e^{-\lambda x}}{\Gamma(\beta + 1)},$$

with $\beta = 2.8$, *diffusion coefficient*

$$c(x) = \frac{x^\alpha \Gamma(1 + \beta - \alpha)}{\Gamma(\beta + 1)},$$

forcing function

$$q(x, t) = e^{-\lambda x - t} \frac{\Gamma(1 + \beta - \alpha)}{\Gamma(\beta + 1)} \left(\frac{(1 - \alpha)\lambda^\alpha x^{\alpha+\beta}}{\Gamma(\beta + 1)} + \frac{\alpha\beta\lambda^{\alpha-1} x^{\alpha+\beta-1}}{\Gamma(\beta)} - \frac{2x^\beta}{\Gamma(1 + \beta - \alpha)} \right)$$

and boundary condition

$$B(t) = u(1, t) = \frac{e^{-\lambda - t}}{\Gamma(\beta + 1)}.$$

The exact solution is given by

$$u(x, t) = \frac{x^\beta e^{-\lambda x - t}}{\Gamma(1 + \beta)}.$$

We solved the equation numerically to time $t = 1$ *with and without extrapolation. The extrapolation was done by halving the grid size to* $\Delta x/2$ *while keeping* Δt *to obtain* $U_{2i,\Delta x/2}$ *and then computing* $U_i = 2U_{2i,\Delta x/2} - U_{i,\Delta x}$. *The maximum error is given in Table 1.*

4 Monte Carlo Simulation

Since the tempered α-stable Lévy process $S_\lambda(t)$ is infinitely divisible, it can be approximated by a random walk at any given mesh Δt. In this section, we provide a general method to generate tempered stable random variables. More generally, the method can be used to simulate any exponentially tempered random variable.

Given a non-negative random variable X with density $f(x)$, define the exponentially tempered density $f_\lambda(x) = e^{-\lambda x}f(x)/\int_0^\infty e^{-\lambda u}f(u)\,du$. Take Y exponentially distributed with mean λ^{-1}, independent of X. Generate (X_i, Y_i) independent and identically distributed with (X, Y), and let $N = \min\{n : X_n \le Y_n\}$. Then X_N has the exponentially tempered density function $f_\lambda(x)$: To see this, let $q = \mathbb{P}(X \le Y) = \int_0^\infty \mathbb{P}(X \le y)\lambda e^{-\lambda y}\,dy$ and compute

$$\begin{aligned}
\mathbb{P}(X_N \le x) &= \sum_{n=1}^{\infty} \mathbb{P}(X_N \le x, N = n) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n \le x, X_n \le Y_n)\mathbb{P}(X_{n-1} > Y_{n-1}) \cdots \mathbb{P}(X_1 > Y_1) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(X_n \le x, X_n \le Y_n)(1-q)^{n-1} \\
&= q^{-1}\mathbb{P}(X_n \le x, X_n \le Y_n)
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\mathbb{P}(X_n \le x, X_n \le Y_n) &= \int_0^\infty \mathbb{P}(X \le x, X \le y)\lambda e^{-\lambda y}\,dy \\
&= \int_0^x \mathbb{P}(X \le y)\lambda e^{-\lambda y}\,dy + \mathbb{P}(X \le x)e^{-\lambda x}
\end{aligned} \tag{32}$$

and hence the density of X_N is

$$\begin{aligned}
\frac{d}{dx}\mathbb{P}(X_N \le x) &= q^{-1}\frac{d}{dx}\left[\int_0^x \mathbb{P}(X \le y)\lambda e^{-\lambda y}\,dy + \mathbb{P}(X \le x)e^{-\lambda x}\right] \\
&= q^{-1}\left[\mathbb{P}(X \le x)\lambda e^{-\lambda x} + f(x)e^{-\lambda x} - \mathbb{P}(X \le x)\lambda e^{-\lambda x}\right]
\end{aligned} \tag{33}$$

which reduces to $f_\lambda(x)$, since $q = \int_0^\infty f(y)e^{-\lambda y}\,dy$ via integration by parts.

If $S(t)$ is stable with index $\alpha < 1$ and its density has Fourier transform $p(k,t) = e^{tc(ik)^\alpha}$ then $S(t) \ge 0$, and the method of [3,20] can be used to simulate $S(t) = Z$: Take U uniform on $[-\pi/2, \pi/2]$, W exponential with mean 1, and set

$$Z = (|c|t)^{1/\alpha}\frac{\sin\alpha(U + \frac{\pi}{2})}{\cos(U)^{1/\alpha}}\left(\frac{\cos(U - \alpha(U + \frac{\pi}{2}))}{W}\right)^{(1-\alpha)/\alpha}. \tag{34}$$

The tempered stable random variable $S_\lambda(t)$ can be simulated by the rejection method: Draw Z, and Y exponential with mean λ^{-1}. If $Y < Z$, reject and draw again; otherwise, set $S_\lambda(t) = Z + ct\alpha\lambda^{\alpha-1}$. To simulate the entire path of the process, write $t = n\Delta t$ and note that $S_\lambda(t) = \sum_{k=1}^n [S_\lambda(k\Delta t) - S_\lambda((k-1)\Delta t)]$, a random walk with all n summands independent and identically distributed with $S_\lambda(\Delta t)$.

When $\alpha > 1$, the stable density $p(x,t)$ with Fourier transform $\hat{p}(k,t) = e^{tc(ik)^\alpha}$ is positive for all $-\infty < x < \infty$ and all $t > 0$; i.e., the support is all of $\mathbb{R}$. The exponentially tempered density $p_\lambda(x,t) = e^{-tc\lambda^\alpha}e^{-\lambda x}p(x,t)$ is still integrable because

$p(x,t) \to 0$ faster than any exponential function as $x \to -\infty$ [28, Theorem 2.5.2 and Eq. (2.5.18-19)]. To simulate these random variables, we adapt the procedure for $\alpha < 1$ and use the following theorem. Recall that for two random variables X, Z with cumulative distribution functions $F(x) = \mathbb{P}(X \le x), G(z) = \mathbb{P}(Z \le z)$, the Kolmogorov-Smirnov distance is defined by $d(X,Z) = \sup\{|F(x) - G(x)| : x \in \mathbb{R}\}$.

Theorem 8 *Let X be a random variable with density $f(x)$ such that the integral $I = \int_{-\infty}^{\infty} e^{-\lambda x} f(x)\,dx$ exists. Define the exponentially tempered density $f_\lambda(x) = e^{-\lambda x} f(x)/I$. Let Y be exponentially distributed with mean λ^{-1}, independent of X. Generate (X_i, Y_i) independent and identically distributed with (X,Y), and for $a \ge 0$ let $N_a = \min\{n : X_n \le Y_n - a\}$. Then the density of X_{N_a} converges in L^1 to f_λ as $a \to \infty$, and hence $d(X_{N_a}, X) \to 0$ in the Kolmogorov-Smirnov distance.*

Proof. Set $q = \mathbb{P}(X \le Y - a)$ and compute $F_a(x) = \mathbb{P}(X_{N_a} \le x) = q^{-1}\mathbb{P}(X \le x, X \le Y - a)$. Note that $F_a(x) = q^{-1}\mathbb{P}(X \le x)$ for $x < -a$, and, similar to (32), for $x \ge -a$ we have

$$F_a(x) = q^{-1}\left(\int_0^{x+a} \mathbb{P}(X \le y - a)\lambda e^{-\lambda y}\,dy + \mathbb{P}(X \le x)e^{-\lambda(x+a)}\right).$$

Then the density of X_{N_a} is $f_a(x) = q^{-1}f(x)$ for $x < -a$ and, by the same argument as (33), $f_a(x) = q^{-1}f(x)e^{-\lambda(x+a)}$ for $x \ge -a$. Note that

$$\begin{aligned} q &= \int_0^\infty P(X \le y - a)\lambda e^{-\lambda y}\,dy = e^{-\lambda a}\int_{-a}^{\infty} P(X \le y)\lambda e^{-\lambda y}\,dy \\ &= \int_{-\infty}^{-a} f(y)\,dy + e^{-\lambda a}\int_{-a}^{\infty} f(y)e^{-\lambda y}\,dy. \end{aligned}$$

Let $h_a = e^{\lambda a}\int_{-\infty}^{-a} f(y)\,dy$ and $g_a = \int_{-\infty}^{-a} e^{-\lambda y} f(y)\,dy$. Then $q = e^{-\lambda a}(I - g_a + h_a)$, $0 \le h_a \le g_a \to 0$ and the L^1 distance

$$\begin{aligned} \int_{-\infty}^{\infty} |f_\lambda(x) - f_a(x)|\,dx &= \int_{-\infty}^{-a}\left|\frac{e^{-\lambda x}f(x)}{I} - \frac{f(x)}{q}\right|dx + \int_{-a}^{\infty}\left|\frac{e^{-\lambda x}f(x)}{I} - \frac{f(x)e^{-\lambda(x+a)}}{q}\right|dx \\ &= \int_{-\infty}^{-a}\left|\frac{e^{-\lambda x}f(x)}{I} - \frac{e^{\lambda a}f(x)}{I - g_a + h_a}\right|dx + \int_{-a}^{\infty}\left|\frac{e^{-\lambda x}f(x)}{I} - \frac{f(x)e^{-\lambda x}}{I - g_a + h_a}\right|dx \\ &\le \frac{2g_a}{I - g_a + h_a} + (I - g_a)\left(\frac{1}{I - g_a + h_a} - \frac{1}{I}\right) \le \frac{3g_a}{I - g_a + h_a} \to 0. \end{aligned}$$

Finally we note that

$$\begin{aligned} d(X_{N_a}, X) &= \sup_{x\in\mathbb{R}} |F_\lambda(x) - F_a(x)| \\ &= \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^{x} f_\lambda(y)\,dy - \int_{-\infty}^{x} f_a(y)\,dy\right| \\ &\le \sup_{x\in\mathbb{R}}\int_{-\infty}^{x} |f_\lambda(y) - f_a(y)|\,dy \\ &= \int_{-\infty}^{\infty} |f_\lambda(y) - f_a(y)|\,dy \le \frac{3g_a}{I - g_a + h_a} \to 0 \end{aligned} \tag{35}$$

which completes the proof $\square$

Note that if $P(X \geq -a) = 1$, then obviously $g_a = 0$ and X_{N_a} has the exponentially tempered function $f_\lambda(x)$ as its density.

To simulate $S_\lambda(t)$ for $\alpha > 1$, choose $a > 0$ so that $P(S(t) < -a)$ is small, draw Z using

$$Z = (ct)^{1/\alpha} \frac{\sin \alpha(U - \frac{\pi}{2})}{\cos(U)^{1/\alpha}} \left(\frac{\cos(U - \alpha(U - \frac{\pi}{2} + \frac{\pi}{\alpha}))}{W} \right)^{(1-\alpha)/\alpha} \tag{36}$$

and Y exponential with mean λ^{-1}. If $Z > Y - a$, reject and draw again; otherwise, set $S_\lambda(t) = Z + ct\alpha\lambda^{\alpha-1}$. Note that a larger a increases accuracy of the simulation method, however a larger a will result in more rejections, increasing computing time. For $\alpha = 1$, use the same method and set $S_\lambda(t) = Z + ct(1 + \ln \lambda)$ where

$$Z = (ct) \left(\ln\left(\frac{\pi}{2} ct\right) + \left(\frac{\pi}{2} + U\right) \tan(U) - \ln\left(\frac{\frac{\pi}{2} W \cos U}{\frac{\pi}{2} + U} \right) \right). \tag{37}$$

A random walk simulation of the particle path can be accomplished as in the case $\alpha < 1$, by simulating and adding independent and identically distributed random jumps $S_\lambda(\Delta t)$. Figure 1 illustrates a random walk simulation of $S_\lambda(t)$ with $\alpha = 1.2$, $c = 1$, $\Delta t = 1$, and $0 \leq t \leq T = 100,000$ for four different values of the tempering parameter λ. The simulation produced no $Z < -2.5$, and the chosen shift $a = 4$ lead to few rejects [$4,000$ out of 100,000 for $\lambda = 0.1$; 350 for $\lambda = 0.01$; 30 for $\lambda = 0.001$; and 5 for $\lambda = 0.0001$]. In this case, the random variable $S(1)$ is stable with index α, mean zero, skewness $+1$, and scale $\sigma = (-\cos(\pi\alpha/2))^{1/\alpha} \approx 0.3578$ in the parameterization of Samorodnitsky and Taqqu [20]. The Kolmogorov-Smirnov bound (35) is less than 10^{-14} (zero to machine precision) which can be verified by numerical integration using the fast efficient codes of Nolan [18] for the stable density. Observe that as λ gets smaller, the anomalous large jumps in Figure 1 increase, leading to superdiffusion. Particle tracking codes can be accomplished by simulating a large number of particles, and generating a histogram of the results at different time points. The particle tracking method can be extended to variable coefficient equations with complicated geometry and boundary conditions.

An alternative simulation method uses the Poisson approximation for an infinitely divisible Lévy process. It follows from results in Cartea and del Castillo-Negrete [2] that the density $p_\lambda(x,t)$ of $S_\lambda(t)$ has Fourier transform

$$\hat{p}_\lambda(k,t) = \exp\left[c_0 t \int_0^\infty (e^{-ikx} - 1 + ikx) e^{-\lambda x} \alpha x^{-\alpha-1}\, dx \right]$$

where $c_0 = c(\alpha - 1)/\Gamma(2 - \alpha)$. Note that this corrects the constant in [2, Eq. (21)]. Given $\varepsilon > 0$ small, we estimate $\hat{p}_\lambda(k,t)$ by

$$\begin{aligned} &\exp\left[c_0 t \int_\varepsilon^\infty (e^{-ikx} - 1 + ikx) e^{-\lambda x} \alpha x^{-\alpha-1}\, dx \right] \\ &= \exp\left[m_\varepsilon c_0 t (\hat{f}_\varepsilon(k) - 1 + ik\mu_\varepsilon) \right] \end{aligned} \tag{38}$$

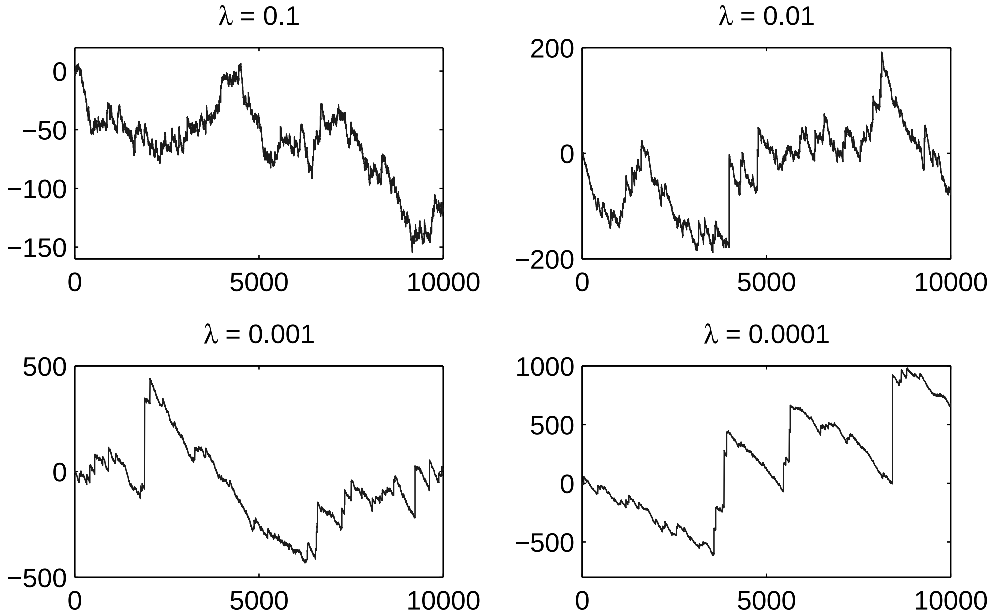

Fig. 1. Tempered stable Lévy motion $S_\lambda(t)$ with $\alpha = 1.2$.

where $m_\varepsilon = \int_\varepsilon^\infty e^{-\lambda x}\alpha x^{-\alpha-1}\,dx$, $f_\varepsilon(x) = m_\varepsilon^{-1}e^{-\lambda x}\alpha x^{-\alpha-1}$ is a probability density, and $\mu_\varepsilon = \int_\varepsilon^\infty x f_\varepsilon(x)\,dx$ is its mean. Recognizing that the last line in (38) is the Fourier transform of a mean-centered compound Poisson [12], we can write

$$S_\lambda(t) \approx \sum_{i=1}^{N_\varepsilon(t)} (J_i - \mu_\varepsilon)$$

where J_i are independent random variables with density f_ε, and $N_\varepsilon(t)$ is a Poisson process with rate $m_\varepsilon c_0$. The approximation becomes exact as $\varepsilon \to 0$. To simulate J_i, use the exponential rejection method for positive random variables X developed in this section, with $P(X > x) = (x/\varepsilon)^{-\alpha}$, so that $X = \varepsilon/U^{1/\alpha}$ with U uniform on $[0, 1]$. To simulate the Poisson process, let $W_i = \ln(U_i)/(m_\varepsilon c_0)$, $T_n = W_1 + \cdots + W_n$, and $N_\varepsilon(t) = \max\{n : T_n \le t\}$. Note that this approximation is actually a continuous time random walk [15,16] with exponentially tempered power-law jumps. Setting $\lambda = 0$ yields a well-known approximation to the anomalous diffusion process $S(t)$. In the special case $\alpha < 1$, a more refined version of this simulation idea was developed in a recent paper of Cohen and Rosiński [5]. Their method reproduces the exact sample path of the process up to small jumps, using a series representation based on a similar Poissonian representation.

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