UNBOUNDED FUNCTIONAL CALCULUS FOR BOUNDED GROUPS WITH APPLICATIONS

BORIS BAEUMER, MARKUS HAASE, AND MIHÁLY KOVÁCS

Abstract. In this paper we develop the unbounded extension of the Hille-Phillips functional calculus for generators of bounded groups. Mathematical applications include the generalised Lévy-Khintchine formula for subordinate semigroups, the analyticity of semigroups generated by fractional powers of group generators, where the power is not an odd integer, and a shifted abstract Grünwald formula. We also give an application of the theory to subsurface hydrology, modeling solute transport on a regional scale using fractional dispersion along flow lines.

1. Introduction

The functional calculus for semigroups goes back already to the seminal work of Hille and Phillips and is today a well-established tool used equally in theory and applications. It is based on the formula

$$f(A) = \int_0^\infty e^{-sA}\,\mu(\mathrm{d}s) \tag{1}$$

where $-A$ generates the semigroup $(e^{-sA})_{s\geq 0}$ and f is the Laplace transform of the bounded measure μ. Already quite early in the development, it was clear that the original Hille-Phillips construction (which yields only bounded operators) should be extended towards unbounded operators, to treat for instance the fractional powers $(A^\alpha)_{0<\alpha\leq 1}$. This was subsequently done by Balakrishnan in [4], and the resulting theory has found many applications, for example in stochastic processes, where Bochner's theory of subordination is a special instance of functional calculus reasoning.

It is obvious and known for a long time that one may exchange the bounded semigroup for a bounded group in the original Hille-Phillips construction. Then one integrates the group $(e^{sA})_{s\in\mathbb{R}}$ against bounded measures on $\mathbb{R}$ and gets a much richer calculus than in the semigroup (the "unilateral") case (for details see Section 2). In [28] the authors considered bounded groups on UMD spaces and proved that the generators of such groups have a bounded bisectorial H^∞-calculus. However, the authors work only with norm estimates and avoid the description of the unbounded calculus in the general case. In [25] an unbounded calculus for general C_0-groups is described and applied, but an explicit examination of the unbounded calculus for *bounded* groups seem to be missing in the literature.

The reason why we now change this comes from applications. In their attempt to model solute transport in subsurface flow, the first and third authors were led to a fractional advection-dispersion equation of the form

$$\frac{\partial}{\partial t}u(t,x) = -v\frac{\partial}{\partial x}u(t,x) + D\left(\frac{\partial}{\partial x}\right)^\alpha u(t,x); u(0,x) = f(x)$$

2000 *Mathematics Subject Classification.* 47A60; 47D03; 26A33 .

Key words and phrases. Subordination; fractional powers; operational calculus; functional calculus; fractional calculus; stable laws; fractional derivatives.

M. Kovács is partially supported by postdoctoral grant No. 623-2005-5078 of the Swedish Research Council (VR).

with $1 < \alpha \le 2$. Since $\alpha > 1$, this equation can no longer be covered by the unilateral theory. However, it turns out that by means of the functional calculus for bounded groups this (and similar equations) can be treated successfully (see Section 5) for more details.

Now, since the time of Hille and Phillips, unbounded functional calculus theory has made some steps forward, mostly triggered by McIntosh's work on sectorial operators and the great success of functional calculus methods in the treatment of maximal regularity questions. In [23] and [24, Chapter 2] the second author has proposed a general scheme of extending a bounded functional calculus to an unbounded one; this scheme is purely algebraic and differs radically from Balakrishnan's more analytical approach in [4]. In our opinion, it is more perspicuous, with the additional advantage that we can cite existing (and not very difficult) theory in place of translating Balakrishnan's (apparently involved) construction to the group setting. Of course, the question how the two approaches relate should be answered (and we do this, see Remark 3.10).

The paper consists of three parts. In the first (Sections 2 and 3) we develop the unbounded functional calculus for a bounded group $(e^{-sA})_{s\in\mathbb{R}}$ in a Banach space X. The main result here is the so-called *transference principle* (Theorem 3.1) which just expresses the original idea of functional calculus: that certain relations between functions imply the corresponding relations between operators. The transference principle has two aspects, one concerning the norm and the other the strong topology. The first is nothing but the (trivial) estimate

$$\|\widehat{\mu}(A)\| \le M \, \|\mu\|_{\mathrm{M}(\mathbb{R})}$$

where μ is a bounded measure on $\mathbb{R}$ and $\|e^{-sA}\| \le M$ for all $s \in \mathbb{R}$. (This is of course highly non-original.) The other aspect concerns the strong topologies (with $\mathrm{M}(\mathbb{R})$ acting on $\mathrm{L}^1(\mathbb{R})$ by convolution). Transference here means that strong convergence $\mu_n \to \mu$ implies strong convergence $\widehat{\mu_n}(A) \to \widehat{\mu}(A)$, and this is at the heart of many useful approximation formulae. The strong transference principle, as it is formulated here, is new, cf. also Remark 3.2. It allows to transfer approximation schemes from functions to operators and hence is important for theoretical numerical analysis.

In the second part (Section 4) we use the developed theory to derive some results about semigroups subordinate to a bounded group (Theorem 4.1), in particular a generalised Levy-Khintchine formula (Theorem 4.4). Furthermore we prove a generation theorem and a Grünwald-type approximation formula for fractional powers A^α with exponents $\alpha > 0$ being different from an odd integer (Theorems 4.6 and 4.9). These extend results of [21] and [50] and are applied in the last part.

The third part consists of a description of the problem in subsurface hydrology that has motivated this research. As a matter of fact, we can only sketch the background here and have to refer to the literature for a more detailed account. Using the theoretical results obtained so far, it follows that the Cauchy problem corresponding to the relevant fractional advection-dispersion equation is well-posed (Theorem 5.1); moreover, based on the Grünwald approximation mentioned earlier, we have set up a numerical example showing the flow around two ellipsoid obstacles, as our model predicts it. Such numerical simulations are important to test the validity of the model.

2. The Extended Hille-Phillips Calculus and the Shift Group

The Laplace transform $\mathcal{L}\mu$ of a bounded measure $\mu \in \mathrm{M}(\mathbb{R}_+)$ is defined by

$$\mathcal{L}\mu(z) := \int_0^\infty e^{-sz}\, \mu(\mathrm{d}s) \qquad (\operatorname{Re} z \ge 0).$$

We write $\mathbb{C}_+ := \{z \in \mathbb{C} \,:\, \operatorname{Re} z \geq 0\}$ and endow

$$\mathcal{E}(\mathbb{C}_+) := \{\mathcal{L}\mu \,:\, \mu \in \mathrm{M}(\mathbb{R}_+)\}$$

and with the norm

$$\|\psi\|_{\mathcal{E}(\mathbb{C}_+)} := \left\|\mathcal{L}^{-1}\psi\right\|_{\mathrm{M}(\mathbb{R}_+)} \qquad (\psi \in \mathcal{E}(\mathbb{C}_+)).$$

Then $\mathcal{E}(\mathbb{C}_+)$ is a Banach algebra with pointwise product, isometrically isomorphic to $\mathrm{M}(\mathbb{R}_+)$ via the mapping $\mathcal{L}$. If $-A$ generates a bounded C_0-semigroup $T = (T(s))_{s\geq 0}$ on a Banach space X, we define as in (1)

$$f(A) := \int_0^\infty T(s)\,\mu(\mathrm{d}s) \qquad (f \in \mathcal{E}(\mathbb{C}_+),\, \mu = \mathcal{L}^{-1}f).$$

Then

$$\Phi_T := (f \longmapsto f(A)) : \mathcal{E}(\mathbb{C}_+) \longrightarrow \mathcal{L}(X)$$

is a well-defined homomorphism of algebras, called the *Hille-Phillips (functional) calculus*.

Although we shall eventually return to the semigroup case (Theorem 4.1), most of the paper will deal with the analogous situation for bounded C_0-*groups*. Let $-A$ generate a bounded C_0-group $(G(s))_{s\in\mathbb{R}}$ on a Banach space X, consider the convolution algebra $\mathrm{M}(\mathbb{R})$, and define

$$G_\mu x := \int_{\mathbb{R}} G(s)x\,\mu(\mathrm{d}s) \qquad (x \in X, \mu \in \mathrm{M}(\mathbb{R})).$$

Similar as above, we may write $G_\mu = f(A)$, where

$$f(z) := (\mathcal{L}\mu)(z) := \int_{\mathbb{R}} e^{-sz}\,\mu(\mathrm{d}s) \qquad (z \in i\mathbb{R})$$

is the Laplace–Stieltjes transform of μ. (Since μ has no support restriction, the Laplace–Stieltjes transform of μ is only defined on $i\mathbb{R}$. Because of the close connection to the Fourier transform, we shall often write $\widehat{\mu}$ instead of $\mathcal{L}\mu$, and $f^\vee$ instead of $\mathcal{L}^{-1}f$.) Define

$$\mathcal{E}(i\mathbb{R}) := \{\widehat{\mu} \,:\, \mu \in \mathrm{M}(\mathbb{R})\} \quad \text{and} \quad \mathcal{E}_0(i\mathbb{R}) := \{\widehat{f} \,:\, f \in \mathrm{L}^1(\mathbb{R})\}.$$

Then we obtain a homomorphism of algebras

$$\Phi_G := (f \longmapsto f(A)) : \mathcal{E}(i\mathbb{R}) \longrightarrow \mathcal{L}(X),$$

which we also call *Hille-Phillips calculus*. (The space $\mathcal{E}_0(i\mathbb{R})$ will be of some importance later.) We shall focus on the group case in the following, but most results hold *mutatis mutandis* in the semigroup case.

The Hille-Phillips calculus can be extended towards an unbounded functional calculus in a canonical way. The method, described at length in [23, 24], is purely algebraic and has nothing to do with groups or semigroups: One starts with a triple $(\mathcal{E}, \mathcal{F}, \Phi)$, where $\mathcal{F}$ is an algebra with unit $\mathbf{1}$, $\mathcal{E}$ is a subalgebra of $\mathcal{F}$ and $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$ is an algebra homomorphism. This situation is called an *abstract functional calculus* (afc). In the situation from above (group case):

$$\mathcal{E} = \mathcal{E}(i\mathbb{R}), \quad \Phi = \Phi_G \quad \text{and} \quad \mathcal{F} = \mathcal{M}(i\mathbb{R})$$

is the algebra of all measurable functions on $i\mathbb{R}$. In the general setting, for $f \in \mathcal{F}$ each member of the set

$$\operatorname{Reg}(f) := \{e \in \mathcal{E} \,:\, ef \in \mathcal{E},\ \Phi(e)\,\text{injective}\}$$

is called a *regulariser* for f. In general, there may be no regularisers at all for a given function f, but if

$$\operatorname{Reg}(\mathbf{1}) \neq \emptyset$$

then the afc is called *proper*. Now, for a proper afc one can extend Φ to the set

$$\mathcal{F}_r := \{f \in \mathcal{F} \,:\, \mathrm{Reg}(f) \neq \emptyset\}$$

which necessarily contains $\mathcal{E}$ and $\mathbf{1}$ by setting

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef)$$

where $e \in \mathrm{Reg}(f)$ is arbitrary. This does not depend on the choice of e and yields a closed (in general unbounded) operator $\Phi(f)$, which coincides with the old $\Phi(f)$ when f itself is in $\mathcal{E}$. The set $\mathcal{F}_r$ is an algebra. If $\mathcal{F}$ is actually an algebra of functions on a set $\Omega \subset \mathbb{C}$ and the function $\iota = (z \longmapsto z)$ is regularisable, then we can form the operator $A := \Phi(\iota)$ and we call the afc a *functional calculus for* A. Instead of $\Phi(f)$ we then write $f(A)$.

Many of the general rules governing the extended mapping Φ can be found in [24, Chapter 1]. However, an important fact is missing there, recently observed by Clark [17, Proposition 3.2].

Proposition 2.1. *Let $(\mathcal{E}, \mathcal{F}, \Phi)$ be a proper abstract functional calculus, and let $f, g \in \mathcal{F}$ be such that there exist $(e_n)_n, (\tilde{e}_n)_n \subset \mathcal{E}$ such that*

(i) $e_n f \in \mathcal{E}$ *for every* $n \in \mathbb{N}$*;*

(ii) $\tilde{e}_n g \in \mathcal{E}$ *for every* $n \in \mathbb{N}$*;*

(iii) $\Phi(e_n) \to I$ *and* $\Phi(\tilde{e}_n) \to I$ *strongly.*

Then

$$\overline{\Phi(f) + \Phi(g)} = \Phi(f+g) \quad \text{and} \quad \overline{\Phi(f)\Phi(g)} = \Phi(fg).$$

Let us come back to where we started. We already have seen that

$$(\mathcal{E}(i\mathbb{R}), \mathcal{M}(i\mathbb{R}), \Phi_G)$$

is an abstract functional calculus. It is proper since for $\mathrm{Re}\,\lambda > 0$ and $\mu = \mathbf{1}_{\mathbb{R}_+}(s)e^{-\lambda s}\mathrm{d}s$ we have

$$\Phi_G\left[(\lambda + z)^{-1}\right] = \int_0^\infty e^{-\lambda s}\, G(s)\,\mathrm{d}s = (\lambda + A)^{-1},$$

which is injective. The function $(z \longmapsto z)$ is regularisable by $(1+z)^{-1}$ and elementary arguments yield that $(z)(A) = A$. As a consequence we obtain $p(A)$ has its usual meaning for every polynomial, and even for every rational function with poles off $\sigma(A)$, see [24, Sect. 1.3].

2.1. **The Shift Group.** The model case of a bounded C_0-group is the right shift group τ on $X_0 := \mathrm{L}^1(\mathbb{R})$ given by

$$(\tau(s)f)(t) := f(t-s) \qquad (s, t \in \mathbb{R}, f \in X_0).$$

Defining $A_0 := d/dt$, then $-A_0$ generates τ (see [24, Sect. 8.4]). With $\psi = \mathcal{L}\mu \in \mathcal{E}(i\mathbb{R})$ an easy calculation shows that

$$\psi(A_0)f = \int_{\mathbb{R}} f(\cdot - s)\,\mu(\mathrm{d}s) = \mu * f \qquad (f \in X_0).$$

Hence $\psi(A_0)$ is the convolution operator induced by $\mu = \mathcal{L}^{-1}\psi$, or equivalently, the "Laplace–Stieltjes" multiplier operator with symbol ψ:

$$\psi(A_0)f = \mathcal{L}^{-1}[\psi \cdot \mathcal{L}f] \qquad (f \in X_0). \tag{2}$$

From this relation it is easy to prove the following.

Lemma 2.2. *Let $\psi \in \mathcal{E}(i\mathbb{R})$. Then $\psi(A_0)$ is injective if and only if $\{\psi \neq 0\}$ is dense in $i\mathbb{R}$.*

The next result is at the heart of the L^1-transference.

Proposition 2.3. *The representation*

$$\Phi_\tau = (\psi \to \psi(A_0)) : \mathcal{E}(i\mathbb{R}) \longrightarrow \mathcal{L}(X_0)$$

(i.e., the Hille-Phillips calculus for d/dt) is isometric.

Proof. Reformulated, the proposition says just that for $\mu \in \mathrm{M}(\mathbb{R})$ the norm of the convolution operator $(f \longmapsto \mu * f)$ on $\mathrm{L}^1(\mathbb{R})$ equals $\|\mu\|_{\mathrm{M}(\mathbb{R})}$. This is a well known result, easily proved by using an approximation of the identity. $\square$

The isometric mapping Φ_τ allows to carry the strong topology over to $\mathcal{E}(i\mathbb{R})$ and $\mathrm{M}(\mathbb{R})$. To be precise, we shall say that a net $(\psi_\alpha)_\alpha \subset \mathcal{E}(i\mathbb{R})$ converges strongly to $\psi \in \mathcal{E}(i\mathbb{R})$ if $\psi_\alpha(A_0)f \to \psi(A_0)f$ for all $f \in X_0$. Analogously, $(\mu_\alpha)_\alpha \subset \mathrm{M}(\mathbb{R})$ is said to converge *strongly* to $\mu \in \mathrm{M}(\mathbb{R})$ if $\mu_\alpha * f \to \mu * f$ for all $f \in \mathrm{L}^1(\mathbb{R})$. Hence $\mu_\alpha \to \mu$ strongly if and only if $\mathcal{L}\mu_\alpha \to \mathcal{L}\mu$ strongly.

The relation (2) will be true for more general functions ψ. To be more precise, let $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$ be any measurable function and define the "Laplace–Stieltjes multiplier operator" M_ψ with symbol ψ and maximal domain $\mathcal{D}(M_\psi)$ by

$$M_\psi f = g \quad :\Longleftrightarrow \quad \psi \cdot \big(\mathcal{L}f\big) = \mathcal{L}g \qquad (f, g \in X_0).$$

If $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$ is regularisable in the Hille-Phillips calculus for A_0 then there is $e \in \mathcal{E}(i\mathbb{R})$ such that $e(A_0)$ is injective and $e\psi \in \mathcal{E}(i\mathbb{R})$ as well. Then, for $f, g \in X_0$,

$$\psi(A_0)f = g \iff (e\psi)(A_0)f = e(A_0)g \iff (e\psi) \cdot (\mathcal{L}f) = e \cdot (\mathcal{L}g).$$

Since $e(A_0)$ is supposed to be injective, the set $\{e \neq 0\}$ is dense, so

$$e \cdot \psi \cdot (\mathcal{L}f) = e \cdot (\mathcal{L}g) \quad \iff \quad \psi \cdot (\mathcal{L}f) = \mathcal{L}g \quad \iff \quad M_\psi f = g.$$

Hence we have shown the second part of the following lemma.

Lemma 2.4. *Let $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$ be a function. Then ψ is regularisable in the Hille-Phillips calculus for $A_0 = d/dt$ if and only if there is $f \in \mathcal{D}(M_\psi)$ such that $\{\mathcal{L}f \neq 0\}$ is dense in $i\mathbb{R}$. In this case one has $\psi(A_0) = M_\psi$.*

Proof. The function ψ is regularisable iff there is a function $e \in \mathcal{E}(i\mathbb{R})$ such that $g := e\psi \in \mathcal{E}(i\mathbb{R})$ and $e(A_0)$ is injective. By multiplying with a function $h \in \mathcal{E}_0(i\mathbb{R})$ such that h has no zeros at all (e.g., h may be a Gaussian), we may suppose without loss of generality that $e, e\psi \in \mathcal{E}_0(i\mathbb{R})$, which means that $\mathcal{L}^{-1}e \in \mathcal{D}(M_\psi)$. $\square$

The following theorem characterises a subclass of regularisable functions.

Theorem 2.5. *Let $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$ be any measurable function. Then the following assertions are equivalent.*

(i) *M_ψ is densely defined.*

(ii) *For every $z \in i\mathbb{R}$ there is $f \in \mathcal{D}(M_\psi)$ such that $(\mathcal{L}f)(z) \neq 0$.*

(iii) *There is $f \in \mathcal{D}(M_\psi)$ such that $(\mathcal{L}f)(z) > 0$ for all $z \in i\mathbb{R}$.*

(iv) *There is $(e_n)_n \subset \mathcal{E}_0(i\mathbb{R})$ such that $e_n \to \mathbf{1}$ strongly, $e_n\psi \in \mathcal{E}_0(i\mathbb{R})$ and $e_n(z) > 0$ for all $z \in i\mathbb{R}, n \in \mathbb{N}$.*

In this case, ψ is continuous and regularisable in the Hille-Phillips calculus for d/dt.

Proof. Let us abbreviate $I := \mathcal{D}(M_\psi)$. Then I is a convolution ideal, i.e., $\mathrm{M}(\mathbb{R}) * I \subset I$. It is obvious that (ii) follows from (i). Suppose that (ii) holds. Then for each $t \in \mathbb{R}$ there is a function $f_t \in I$ such that $\widehat{f_t}(it) \neq 0$. Since I is a convolution ideal we may pass to $f_t * f_t^*$ with $f_t^*(s) := \overline{f_t(-s)}$ for $s \in \mathbb{R}$ and hence $\widehat{f_t} \geq 0$ without loss of generality. By compactness, for each interval $[a, b] \subset \mathbb{R}$ one finds a function $f \in I$ such that $\widehat{f} \geq 0$ and $\widehat{f}(it) > 0$ for all $t \in [a, b]$. Choose such a function $f_n \in I$

for the interval $[-n, n]$, $n \in \mathbb{N}$, and let $g_n \in \mathrm{L}^1(\mathbb{R})$ be such that $\widehat{g_n} = \psi \widehat{f_n}$. Choose $\alpha_n > 0$ in such a way that

$$\sum\nolimits_{n\in\mathbb{N}} \alpha_n \left(\|f_n\|_1 + \|g_n\|_1\right) < \infty,$$

and define $f := \sum_{n\in\mathbb{N}} \alpha_n f_n$ and $g := \sum_{n\in\mathbb{N}} \alpha_n g_n$. Then

$$\widehat{g} = \sum\nolimits_{n\in\mathbb{N}} \alpha_n \widehat{g_n} = \sum\nolimits_{n\in\mathbb{N}} \alpha_n \widehat{f_n} \psi = \widehat{f} \psi.$$

Hence $f \in I$ and $\widehat{f} = \sum_{n\in N} \alpha_n \widehat{f_n} > 0$ everywhere, and (iii) is established.

Now suppose that (iii) holds. Then from Wiener's Tauberian theorem [43, Theorem 9.4] it follows that I is dense. Let $(\phi_n)_n$ be any approximate identity in $\mathrm{L}^1(\mathbb{R})$. Since I is dense, we can find $g_n \in I$ such that $\|g_n - \phi_n\|_1 \to 0$; clearly, also $(g_n)_n$ is an approximate identity, and so without loss of generality we may suppose that $\phi_n \in I$. Because

$$\phi_n^* * h = (\phi_n * h^*)^* \to (h^*)^* = h$$

for every $h \in \mathrm{L}^1(\mathbb{R})$, also $(\phi_n^*)_n$ is an approximate identity of $\mathrm{L}^1(\mathbb{R})$ and since I is an ideal, $\phi_n * \phi_n^* \in I$. But then

$$\eta_n := (\phi_n * \phi_n^*) \; + \; n^{-1} f$$

is an approximate identity with $e_n(it) := \widehat{\eta_n}(it) = |\widehat{\phi_n}(it)|^2 + n^{-1}\widehat{f}(it) > 0$ for all $t \in \mathbb{R}$.

Finally, to see that (iv) implies (i) simply note that each e_n is a regulariser for ψ, so $e_n(A_0)f \in \mathcal{D}(A_0) = \mathcal{D}(M_\psi)$ and $e_n(A_0)f \to f$ as $n \to \infty$, for every $f \in \mathrm{L}^1(\mathbb{R})$. □

In the next section we shall see that the shift group is a model case for general groups. For example, we shall see that a function ψ satisfies the equivalent conditions of Theorem 2.5 if and only if ψ is regularisable within the Hille-Phillips calculus of each bounded group.

3. The Transference Principle and Unbounded Functional Calculus

In this section we shall transfer results about the functional calculus for $A_0 = d/dt$ on $\mathrm{L}^1(\mathbb{R})$ to results about that for A, whenever $-A$ generates an arbitrary bounded group on a Banach space X.

Theorem 3.1. (Fundamental Theorem of Transference)
The Hille-Phillips calculus for $A_0 = d/dt$ on $X_0 = \mathrm{L}^1(\mathbb{R})$

$$\Phi_\tau := [\psi \longmapsto \psi(A_0)] : \mathcal{E}(i\mathbb{R}) \longrightarrow \mathcal{L}(X_0)$$

is isometric, and also a homeomorphism onto its image with respect to the strong topologies. Let $-A$ generate a bounded C_0-group on a Banach space X. Then the Hille-Phillips calculus

$$\Phi_G = [\psi \longmapsto \psi(A)] : \mathcal{E}(i\mathbb{R}) \longrightarrow \mathcal{L}(X)$$

is continuous with respect to the norm topologies with

$$\|\psi(A)\| \le [\sup_{s\in\mathbb{R}} \|G(s)\|] \;\; \|\psi\|_{\mathcal{E}(i\mathbb{R})} \qquad (\psi \in \mathcal{E}(i\mathbb{R})).$$

Furthermore, on norm bounded sets Φ_G is continuous with respect to the strong topologies.

Proof. The first part is simply Proposition 2.3 and the definition of the strong topology on $\mathcal{E}(i\mathbb{R})$. The estimate in the second part is trivial. So suppose that one has a norm-bounded net $\psi_\alpha = \mathcal{L}\mu_\alpha$, with $\psi_\alpha \to 0$ strongly. Take $\eta = \mathcal{L}f$ for some $f \in X_0$. Then

$$\|\psi_\alpha \eta\|_{\mathcal{E}(i\mathbb{R})} = \|\mathcal{L}(\mu_\alpha * f)\|_{\mathcal{E}(i\mathbb{R})} = \|\mu_\alpha * f\|_{\mathrm{L}^1} \to 0.$$

Hence $\psi_\alpha(A)\eta(A) = (\psi_\alpha\eta)(A) \to 0$ in norm. This shows that $\psi_\alpha(A) \to 0$ strongly on

$$\mathrm{span}\{\eta(A)x \,:\, \eta \in \mathcal{E}(i\mathbb{R}), x \in X\}.$$

Since this set is dense in X (look at $\eta_h(z) = (1+hz)^{-1}$, $h \searrow 0$), we may conclude that $\psi_\alpha(A) \to 0$ strongly. □

Remark 3.2. In their famous and influential monograph [18], Coifman and Weiss coined the name 'transference principle'. Simplifying a little, their result allows to estimate the norms of averages over general bounded C_0-groups acting on an arbitrary L^p-space by the norms of the associated convolution operators on $\mathrm{L}^p(\mathbb{R})$. The main strength of this theorem lies in the fact that $1 < p < \infty$ is allowed, the estimate in the case $p = 1$ being essentially trivial. However, the Coifman–Weiss theory does not extend to strong convergence, which is the main point in Theorem 3.1.

Corollary 3.3. *Let $-A$ be the generator of a bounded C_0-group, and let $\psi \in \mathcal{E}(i\mathbb{R})$.*

a) **(Transference of approximate identities)** *Let $\psi \in \mathcal{E}(i\mathbb{R})$ such that $\psi(0) = 1$. Then $\psi(hA) \to I$ strongly as $h \searrow 0$.*

b) *If $\mathcal{R}(\psi(A_0))$ is dense, then $\mathcal{R}(\psi(A))$ is dense.*

Proof. a) It is easy to show that $\psi(hz)(A) = \psi(hA)$ for $h \geq 0$. By virtue of Theorem 3.1 it suffices to show the result for $A = A_0$. The proof is easy, using that $\mathrm{C_c}(\mathbb{R})$ is dense in $\mathrm{L}^1(\mathbb{R})$ (cf. [14, Thm. 3.1.6]).
b) Let $f \in X_0$ be arbitrary. By hypothesis, there is $f_n \in X_0$ such that $\psi(A_0)f_n \to f$. Taking Laplace-Stieltjes transforms and inserting A yields $\psi(A)(\mathcal{L}f_n)(A) \to (\mathcal{L}f)(A)$ in norm. This shows that $\{e(A)x \,:\, e \in \mathcal{E}_0(i\mathbb{R}), x \in X\}$ is contained in the closure of the range of $\psi(A)$. Using an approximate identity, one sees that the former space is dense. □

We now turn to the extended functional calculi.

Lemma 3.4. *Let $e \in \mathcal{E}_0(i\mathbb{R})$. Then the following assertions are equivalent:*

(i) *$e(z) \neq 0$ for all $z \in i\mathbb{R}$.*

(ii) *Whenever $-A$ generates a bounded C_0-group on a Banach space, $e(A)$ is injective.*

(iii) *Whenever $-A$ generates a bounded C_0-group on a Banach space, $e(A)$ has dense range.*

Proof. (i) implies (ii): Fix $f \in \mathrm{L}^1(\mathbb{R})$ such that $\mathcal{L}f = e$. Suppose that for some $x \in X$

$$\int_{\mathbb{R}} G(s)x\, f(s)\,\mathrm{d}s = e(A)x = 0.$$

Multiplying by $G(t)$ for $t \geq 0$, we see that $G(\cdot)x * f^{\sim} = 0$ on $\mathbb{R}_+$, where $f^{\sim}(s) := f(-s)$, $s \in \mathbb{R}$. By Wiener's theorem, $G(\cdot)x * f^{\sim} = 0$ on $\mathbb{R}_+$ for *all* $f \in \mathrm{L}^1(\mathbb{R})$ and by using an approximation of the identity, $x = G(0)x = 0$.
(ii) implies (i): Fix $t \in \mathbb{R}$ and consider the special case of the group $G(s) = e^{-ist}$ on $\mathbb{C}$. Then $-A = -it$ is the generator and $0 \neq e(A) = e(it)$.
(i) implies (iii): By Wiener's theorem, (i) implies that $e(A_0)$ has dense range, and this implies (iii) by Corollary 3.3.
(iii) implies (i): If (i) is not satisfied, then $\mathcal{R}(e(A_0))$ cannot be dense, by (the trivial part of) Wiener's theorem. □

Corollary 3.5. *Let $-A$ generate a bounded C_0-group on a Banach space X, and let $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$. If ψ is regularisable within the extended Hille-Phillips calculus for A, then there is a regulariser e for ψ such that $e, e\psi \in \mathcal{E}_0(i\mathbb{R})$.*

Proof. Let $\tilde{e} = \mathcal{L}\mu$ be any regulariser for ψ. Take $h \in \mathrm{L}^1(\mathbb{R})$ such that $\eta(z) := (\mathcal{L}h)(z) > 0$ for all $z \in i\mathbb{R}$. Then by the previous lemma $\eta(A)$ is injective. Hence $e := \eta\tilde{e} \in \mathcal{E}_0(i\mathbb{R})$ is a regulariser for ψ, and $e\psi = \eta(\tilde{e}\psi) \in \mathcal{E}_0(i\mathbb{R})$ as well. □

We can now extend Theorem 2.5

Theorem 3.6. *For $\psi : i\mathbb{R} \longrightarrow \mathbb{C}$ the following assertions are equivalent.*

(i) *M_ψ is densely defined, i.e. ψ satisfies the equivalent conditions of Theorem 2.5.*

(ii) *$\psi(A)$ is defined via the extended Hille-Phillips calculus whenever $-A$ generates a bounded C_0-group on a Banach space.*

Suppose that (i) and (ii) hold. Then there is a sequence $(e_n)_n \subset \mathcal{E}_0(i\mathbb{R})$ such that

a) *e_n is a regulariser for ψ in the Hille-Phillips calculus for A;*

b) *$e_n(A) \to I$ strongly as $n \to \infty$;*

whenever $-A$ generates a bounded C_0-group on a Banach space.

Proof. Suppose that (i) holds. Then by Theorem 2.5 there is $(e_n)_n \subset \mathcal{E}_0(i\mathbb{R})$ such that $e_n \to \mathbf{1}$ strongly, $e_n\psi \in \mathcal{E}_0(i\mathbb{R})$ and $e_n(z) > 0$ for all $z \in i\mathbb{R}, n \in \mathbb{N}$. Let $-A$ generate a bounded C_0-group. By Lemma 3.4, e_n is a regulariser for ψ in the Hille-Phillips calculus for A and by Theorem 3.1, part b), $e_n(A) \to I$ strongly. This proves (ii), a) and b).
Conversely, suppose that (ii) holds. Consider the *left* shift group on $\mathbf{BUC}(\mathbb{R})$ with generator $-A$. By Corollary 3.5 we find a regulariser $e \in \mathcal{E}_0(i\mathbb{R})$ for ψ, such that $e\psi \in \mathcal{E}_0(i\mathbb{R})$. Writing $\eta := \mathcal{L}^{-1}e \in X_0$ we have that $\eta \in \mathcal{D}(M_\psi)$. By Theorem 2.5 (iii) it suffices to show that $e(z) \neq 0$ for every $z \in i\mathbb{R}$. It is easy to see that

$$\langle e(A)f, g\rangle = \langle f, e(A_0)g\rangle \qquad (f \in \mathbf{BUC}(\mathbb{R}), g \in X_0).$$

Fix $z \in i\mathbb{R}$. Since $f := e^{-tz}$ is not zero and $e(A)$ is injective, $e(A)f \neq 0$ as well, hence we find $g \in X_0$ such that

$$0 \neq \langle e(A)f, g\rangle = \int_{\mathbb{R}} e^{-tz}\, (e(A_0)g)(t)\,\mathrm{d}t = \mathcal{L}(e(A_0)g)(z) = e(z)(\mathcal{L}g)(z).$$

Consequently, $e(z) \neq 0$. □

Let us define

$$\mathcal{F} := \{\psi : i\mathbb{R} \longrightarrow \mathbb{C} \,:\, \mathcal{D}(M_\psi) \text{ is dense in } \mathrm{L}^1(\mathbb{R})\}.$$

By Theorem 2.5 and Lemma 2.4 if $\psi \in \mathcal{F}$ then $M_\psi = \psi(A_0)$; and by Theorem 3.6, $\psi(A)$ is defined whenever $-A$ generates a a bounded C_0-group on a Banach space.

To see that $\mathcal{F}$ is a rich set of functions, recall Carlson's inequality [16] (see also [1, Lemma 8.2.1]), which in our terminology says that there is a constant C such that whenever $\phi \in \mathrm{W}^{1,2}(i\mathbb{R})$ then $\phi \in \mathcal{E}_0(i\mathbb{R})$ and

$$\|\phi\|_{\mathcal{E}_0(i\mathbb{R})} \leq C\, \|\phi\|^{1/2}_{\mathrm{L}^2(i\mathbb{R})}\, \|\phi'\|^{1/2}_{\mathrm{L}^2(i\mathbb{R})}\,. \tag{3}$$

Using this it is not difficult to see that

$$\{\psi \in \mathrm{C}(i\mathbb{R}) : \psi' \in \mathrm{L}^2_{loc}(i\mathbb{R})\} \subset \mathcal{F} \subset \mathrm{C}(i\mathbb{R}).$$

The following is a consequence of Theorems 2.5 and 3.6, Proposition 2.1, and general functional calculus theory.

Corollary 3.7. *Let $\phi, \psi \in \mathcal{F}$. Then $\phi\psi, \phi + \psi \in \mathcal{F}$ and*

$$\overline{\phi(A) + \psi(A)} = (\phi + \psi)(A) \quad \text{and} \quad \overline{\phi(A)\psi(A)} = (\phi\psi)(A)$$

whenever $-A$ generates a bounded C_0-group on a Banach space X.

The transference theorem can be employed to transfer certain relations between the unbounded operators $\psi(A_0) = M_\psi$ to the operators $\psi(A)$.

Proposition 3.8. *Suppose $\phi, \psi \in \mathcal{F}$ and let $-A$ generate a bounded C_0-group on a Banach space X. Then the following statements hold:*

a) $\sigma(\psi(A)) \subset \sigma(M_\psi)$ *(Spectral inclusion).*

b) *If $\mathcal{R}(M_\phi) \subset \mathcal{D}(M_\psi)$ and $\phi \in \mathcal{E}(i\mathbb{R})$, then $\mathcal{R}(\phi(A)) \subset \mathcal{D}(\psi(A))$.*

c) *If $M_\phi + M_\psi = M_{\phi+\psi}$ and $\varrho(M_{\phi+\psi}) \neq \emptyset$, then $\phi(A) + \psi(A) = (\phi + \psi)(A)$.*

d) *If $M_\phi M_\psi = M_{\phi\psi}$ and $\varrho(M_{\phi\psi}) \neq \emptyset$, then $\phi(A)\psi(A) = (\phi\psi)(A)$.*

Proof. We only give a sketch. a) Let $\lambda \in \varrho(M_\psi) = \varrho(\psi(A_0))$. Then $f(A_0) \in \mathcal{L}(X_0)$, where $f := (\lambda - \psi)^{-1}$. Hence $f(A) \in \mathcal{L}(X)$, by transference, and so $\lambda \in \varrho(\psi(A))$.
b) By hypotesis $(\psi\phi)(A_0) = \psi(A_0)\phi(A_0) = M_\psi M_\phi \in \mathcal{L}(X_0)$. By transference $(\psi\phi)(A) \in \mathcal{L}(X)$, hence $\psi(A))\phi(A) = (\psi\phi)(A)$ is bounded.
c) Define $\eta := \phi + \psi$. By hypothesis, $\psi(A_0) + \phi(A_0) = \eta(A_0)$. This yields $\mathcal{D}(\psi(A_0)) = \mathcal{D}(\phi(A_0)) = \mathcal{D}(\eta(A_0)) = \mathcal{R}((\lambda - \eta)^{-1}(A_0))$, for some $\lambda \in \mathbb{C}$. Now apply a) and b). d) is similar to c). $\square$

Finally, we transfer approximations of unbounded operators from the model case L^1 to general bounded groups:

Proposition 3.9. *Let $(\psi_n)_n \subset \mathcal{F}$ and suppose that there exists $\phi \in \mathcal{F}$ with $\mathcal{D}(M_\phi) \subset \bigcap_n \mathcal{D}(M_{\psi_n})$ and such that $M_{\psi_n} f$ converges for all $f \in \mathcal{D}(M_\phi)$. Then there is a unique $\psi \in \mathcal{F}$ such that $\psi_n \to \psi$ pointwise. Moreover, $\mathcal{D}(M_\phi) \subset \mathcal{D}(M_\psi)$ and, if $\varrho(M_\phi) \neq \emptyset$,*

$$\psi_n(A)x \to \psi(A)x \qquad (x \in \mathcal{D}(\phi(A)))$$

whenever $-A$ generates a bounded C_0-group on a Banach space X.

Proof. By hypothesis, $\psi_n f$ is norm convergent in $\mathcal{E}(i\mathbb{R})$, for all f from a dense subset of $\mathcal{E}_0(i\mathbb{R})$. By varying f we see that $\psi := \lim_n \psi_n$ exists pointwise, and it follows readily that $\psi \in \mathcal{F}$ and $\mathcal{D}(M_\phi) \subset \mathcal{D}(M_\psi)$. If in addition $\lambda \in \varrho(M_\phi)$, let $e := (\lambda - \phi)^{-1} \in \mathcal{E}(i\mathbb{R})$. Then $\psi_n e \to \psi e$ strongly. Inserting A yields $\psi_n(A)e(A) \to \psi(A)e(A)$ strongly. But clearly $e(A) = R(\lambda, \phi(A))$. $\square$

Remark 3.10. As mentioned in the introduction, Balakrishnan [4] has constructed an unbounded extension of the Hille-Phillips calculus for bounded *semigroups*. Going through his paper one may convince oneself that the same construction would work *mutatis mutandis* for bounded groups. This would yield a functional calculus for the class $\mathcal{F}$ defined above, and this — by Theorem 3.6 — would coincide with our calculus restricted to that class. For more detail, see [26].

4. Mathematical Applications

We shall illustrate the effectivity of the developed functional calculus by looking at three examples. First, we shall transfer regularity and stability properties of convolution semigroups to semigroups obtained by subordination and in case of a positive subordinator we transfer the Lévy-Khintchine formula to the subordinate semigroup. Next we show that certain fractional powers of group generators generate analytic semigroups and finally we prove a Grünwald type approximation formula for fractional powers.

4.1. **Subordination.** Suppose that $S = (\mu_t)_{t \geq 0} \subset \mathrm{M}(\mathbb{R})$ is a uniformly bounded convolution semigroup with $\mu_0 = \delta_0$. By virtue of Theorem 3.1 we are allowed to identify S with the induced convolution operators on $X_0 = \mathrm{L}^1(\mathbb{R})$. Hence we often write

$$S(t) = [\mathcal{L}\mu_t](A_0) \in \mathcal{L}(X_0) \qquad (t \geq 0)$$

and this is a bounded semigroup. We shall suppose in the following that this semigroup is strongly continuous. Under this (and, in fact, weaker) condition, one finds that

$$\mathcal{L}\mu_t = e^{-t\psi} \qquad (t \geq 0)$$

for some function $\psi : i\mathbb{R} \longrightarrow \mathbb{C}_+$, the so-called *log-characteristic function*. It is easy to see that $-M_\psi = -\psi(A_0)$ is the generator of S (see also the proof of Theorem 4.1 below).

Now, given any bounded group G on a Banach space X, with generator $-A$, we can form

$$S_G(t) := [e^{-t\psi}](A) = \int_{\mathbb{R}} G(s)\, \mu_t(\mathrm{d}s) \qquad (t \geq 0),$$

and this is a bounded semigroup, by functional calculus. Following Bochner [11, 12], we call S the *subordinator* and S_G the *subordinate semigroup*. We shall apply the transference principle of the previous section to transfer properties from S to S_G.

Theorem 4.1. *The semigroup S_G is strongly continuous and has generator $-B = -\psi(A)$. For every $\phi \in \mathcal{E}(\mathbb{C}_+)$ one has*

$$\phi(B) = (\phi \circ \psi)(A) \qquad \textit{(Composition Rule)}$$

and the transference estimate

$$\|\phi(B)\|_{\mathcal{L}(X)} \leq \big[\sup_{s\in\mathbb{R}} \|G(s)\|\,\big]\, \|\phi(M_\psi)\|\,. \tag{4}$$

Furthermore, if $(\phi_n)_{n\in\mathbb{N}} \subset \mathcal{E}(\mathbb{C}_+)$ and $\phi_n(M_\psi) \to 0$ strongly on $\mathrm{L}^1(\mathbb{R})$, then $\phi_n(B) \to 0$ strongly on X.

Proof. Theorem 3.1 implies that the strong continuity of $t \longmapsto S(t) = [e^{-t\psi}](A_0)$ carries over to that of $t \longmapsto S_G(t) = [e^{-t\psi}](A)$. Denote the generator of S, S_G by $-B_0, -B$, respectively, and let $\nu := \mathcal{L}^{-1}\phi \in \mathrm{M}(\mathbb{R}_+)$. Then

$$\begin{aligned}(\phi\circ\psi)(A_0) &= \left[\int_0^\infty (e^{-s\psi})\,\nu(\mathrm{d}s)\right](A_0) = \int_0^\infty (e^{-s\psi})(A_0)\,\nu(\mathrm{d}s)\\ &= \int_0^\infty S(s)\,\nu(\mathrm{d}s) = \phi(B_0).\end{aligned}$$

Here we used the first part of Theorem 3.1 and observe that the integral

$$\int_0^\infty (e^{-s\psi})\,\nu(\mathrm{d}s)$$

converges within a norm-bounded subset of $\mathcal{E}(i\mathbb{R})$ with respect to the strong topology. Hence applying the second part of Theorem 3.1 we may replace A_0, B_0 by A, B in the computation above. Specialising $\phi = (1+z)^{-1}$ we see that indeed $\psi(A) = B$. □

Corollary 4.2. *If for S one of the following properties is true, then the same property holds also for S_G:*

a) $\sup_{t\geq 0} e^{-\omega t}\,\|S(t)\| < \infty$.
b) *S is exponentially stable.*
c) *S is strongly stable.*
d) *S is bounded holomorphic of angle $\theta \in (0, \pi/2]$.*
e) *S is differentiable for $t > t_0$.*
f) *S is norm continuous for $t > t_0$.*
g) *S is uniformly continuous; i.e, S has a bounded generator.*

Proof. For a) apply the transference estimate (4) to $\phi(z) = e^{-tz}e^{\omega t}$, for f) to $\phi(z) = e^{-tz} - e^{-t_0 z}$, and for g) to $\phi(z) = e^{-tz} - 1$. For b) use the last part of Theorem 4.1 with $\phi_n(z) = e^{-t_n z}(1+z)^{-1}e^{\omega t_n}$, and for c) with $\phi_n(z) = e^{-t_n z}$ where $t_n \to \infty$. To prove d), notice that since S is bounded holomorphic of angle $\theta \in (0, \pi/2]$ it follows that the sectorial region $\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \theta\} \setminus \{0\} \subset \rho(-M_\psi)$ and hence $\Sigma_\theta \subset \rho(-B)$ by Proposition 3.8.a). Furthermore, $\|R(\lambda, -\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}))} \le \frac{M_\varepsilon}{|\lambda|}$ for all $\lambda \in \Sigma_{\theta-\varepsilon}$ and $\varepsilon \in (0, \delta)$ for some $M_\varepsilon \ge 1$. By the transference estimate (4),

$$\left\| \sum_{k=1}^{n} a_k R^k(\lambda, -B) \right\|_{\mathcal{B}(X)} \le M_G \left\| \sum_{k=1}^{n} a_k R^k(\lambda, -\mathcal{M}_\psi) \right\|_{\mathcal{B}(L_1(\mathbb{R}))}, \tag{5}$$

for all $\operatorname{Re}\lambda > 0$, $a_k \in \mathbb{C}, n \in \mathbb{N}$. By the continuity of the resolvents, this holds for $\lambda \in i\mathbb{R}$ as well. Let $\lambda \in \Sigma_{\theta-\varepsilon}$. Then there is $\lambda_0 \in i\mathbb{R}$ and $r > 0$ such that $\lambda \in D(\lambda_0, r) \subset \rho(-\mathcal{M}_\psi) \subset \rho(-B)$, where $D(\lambda_0, r)$ is the open disc with centre λ_0 and radius r. Thus, the Taylor series expansion of the resolvents on $D(\lambda_0, r)$ and the transference estimate (5) yield $\|R(\lambda, -B)\|_{\mathcal{B}(L_1(\mathbb{R}))} \le \frac{M_G M_\varepsilon}{|\lambda|}$. For the proof of e) note that S is differentiable for $t > t_0$ if and only if $(\psi e^{-t\psi})(A_0) = \psi(A_0)(e^{-t\psi})(A_0) \in \mathcal{L}(X_0)$ for $t > t_0$. Transference carries this over to A. □

We remark that the transference of analyticity for subordinate semigroups and measures μ_S supported on $[0, \infty)$ was proved by Carasso and Kato in [15] while the transference of the angle in the same setup, but for a restricted class of measures, was proved by Berg, Boyadzhiev and de Laubenfels in [9].

Finally we formulate a result that allows the construction of convergent numerical approximations to the solution of the Cauchy problem

$$\dot{u}(t) = -\psi(A)u(t), \qquad u(0) = x \in X$$

based on convergent numerical approximations to solutions of

$$\dot{u}(t) = -\mathcal{M}_\psi u(t), \qquad u(0) = f \in L_1(\mathbb{R}).$$

Proposition 4.3. *Let $(\psi_n) \subset \mathcal{F}$ and $\phi, \psi \in \mathcal{F}$ as in Proposition 3.9. Suppose that each $-M_{\psi_n}$ generates a C_0-semigroup S_n, such that* $\sup_{n,s} \|S_n(s)\| < \infty$. *If for some $\lambda \in \mathbb{R}$, $\lambda - M_\psi$ has dense range, then also $-M_\psi$ generates a bounded C_0-semigroup S. Furthermore, whenever $-A$ generates a bounded C_0-group on a Banach space X,*

$$G_{S_n}(s)x \to G_S(s)x \qquad (x \in X)$$

uniformly in $s \ge 0$ on compact intervals.

Proof. Employing Theorem 2.5, part (iv), one easily shows that $\mathcal{D}(M_\phi)$ is a core for M_ψ. Then the Trotter–Kato theorem [19, Chapter III, Theorems 4.8 and 4.9] implies that $-M_\psi$ generates a bounded C_0-semigroup on $\mathrm{L}^1(\mathbb{R})$, and $R(\lambda, M_{\psi_n}) \to R(\lambda, M_\psi)$ strongly for all $\lambda < 0$. This transfers to the strong convergence

$$R(\lambda, \psi_n(A))x \to R(\lambda, \psi(A))x \qquad (x \in X,\, \lambda < 0).$$

Moreover, $-\psi_n(A)$ and $-\psi(A)$ generate the subordinate semigroups G_{S_n} and G_S, respectively, which are uniformly bounded by $M \sup_s \|G(s)\|$ (see above). Applying the Trotter–Kato theorem again but now on X, the proof is complete. □

4.1.1. *The generalised Lévy-Khintchine formula.* Of special interest to applications is the case where the subordinator S is positive; i.e., $S(t)f \ge 0$ for all $f, t \ge 0$. Then μ_S^t has a Lévy-Khintchine representation (see, e.g. [29, Thm. 23.13.1] or [30]), namely ψ is given by

$$\psi(z) = c + dz - \frac{\sigma^2}{2}z^2 + \int_{s\neq 0} \left(1 - e^{-zs} - \frac{zs}{1+s^2}\right)\phi(\mathrm{d}s), \tag{6}$$

where the Lévy measure ϕ — which satisfies $\int_{s\neq 0} \min\{1, s^2\}\phi(\mathrm{d}s) < \infty$ — and the constants $c \geq 0$, $d \in \mathbb{R}$ and $\sigma^2 \geq 0$ are uniquely determined. If $\|S(s)\| = 1$ for all $s \geq 0$, subordination has a stochastic interpretation as randomising time against an infinitely divisible distribution and ψ is given by the Lévy-Khintchine formula above with $c = 0$. The following theorem in the special case $A = A_0$, is [29, Thm. 23.14.2] and it is also proved in [2] in the multiparameter case, but not with functional calculus techniques.

Theorem 4.4. *Let S be positive; i.e., the log-characteristic function is given by the Lévy-Khintchine formula* (6)*. Then $\mathcal{D}(A^2) \subset \mathcal{D}(\psi(A))$ and*

$$A_S x = cx + dAx - \frac{\sigma^2}{2}A^2 x + \int_{s\neq 0}\Big(x - G(s)x - \frac{sAx}{1+s^2}\Big)\,\phi(\mathrm{d}s) \qquad (x \in \mathcal{D}(A^2)),$$

whenever $-A$ generates a bounded C_0-group on a Banach space X and $-A_S$ is the generator of the subordinate semigroup G_S.

Proof. Let us define $\nu(\mathrm{d}s) := (s^2/(1+s^2))\phi(\mathrm{d}s) \in \mathrm{M}(\mathbb{R}\setminus\{0\})$. We first split ψ into “good” and “bad” parts:

$$\begin{aligned}
\psi_0(z) &:= c + dz - \frac{\sigma^2}{2}z^2,\\
\psi_1(z) &:= \int_{s\neq 0}(1-e^{-sz})\left(\frac{s^2}{1+s^2}\right)\phi(\mathrm{d}s) = \int_{s\neq 0}(1-e^{-sz})\,\nu(\mathrm{d}s),\\
\psi_2(z) &:= \int_{s\neq 0}\frac{1-e^{-sz}-sz}{s^2}\left(\frac{s^2}{1+s^2}\right)\phi(\mathrm{d}s) = \int_{s\neq 0}\frac{1-e^{-sz}-sz}{s^2}\,\nu(\mathrm{d}s).
\end{aligned}$$

Then $\psi = \psi_0 + \psi_1 + \psi_2$. By Hille-Phillips calculus,

$$\psi_1(A) = \int_{s\neq 0}(I - G(s))\,\nu(\mathrm{d}s) \in \mathcal{L}(X)$$

and by general functional calculus theory, $\psi_0(A) = cI + dA - \frac{\sigma^2}{2}A^2$. Now,

$$\frac{1-e^{-sz}-sz}{s^2} = -\frac{z^2}{s^2}\int_0^s\int_0^r e^{-tz}\,\mathrm{d}t\,\mathrm{d}r = -\frac{z^2}{s^2}\int_0^s (s-r)e^{-rz}\,\mathrm{d}r = z^2(\mathcal{L}f_s)(z)$$

with $f_s = -s^{-2}\mathbf{1}_{(0,s)}(s-\cdot)$ for $s > 0$ and $f_s = s^{-2}\mathbf{1}_{(s,0)}(s-\cdot)$ for $s < 0$. General functional calculus theory yields

$$A^2[\mathcal{L}f_s](A) = [z^2\mathcal{L}f_s](A) = \left(\frac{1-e^{-sz}-sz}{s^2}\right)(A) = \frac{I - G(s) - sA}{s^2}$$

for $s \neq 0$. It is easily seen that the mapping $(s \longmapsto f_s) : \mathbb{R}\setminus\{0\} \longrightarrow \mathrm{L}^1$ is continuous. Let $\mu \in \mathrm{M}(\mathbb{R})$ be such that $\mathcal{L}\mu = z^2(1+z)^{-2}$. Then by transference

$$\begin{aligned}
\psi_2(A)(1+A)^{-2} &= \left(\frac{\psi_2(z)}{(1+z)^2}\right)(A) = \left(\frac{\psi_2(z)}{z^2}\frac{z^2}{(1+z)^2}\right)(A)\\
&= \left(\int_{s\neq 0}\mathcal{L}[f_s * \mu]\,\nu(\mathrm{d}s)\right)(A) = \int_{s\neq 0}(z^2[\mathcal{L}f_s](1+z)^{-2})(A)\,\nu(\mathrm{d}s)\\
&= \int_{s\neq 0}\left(z^2[\mathcal{L}f_s]\right)(A)(1+A)^{-2}\,\nu(\mathrm{d}s)\\
&= \int_{s\neq 0}\left(\frac{1-e^{-sz}-sz}{s^2}\right)(A)(1+A)^{-2}\,\nu(\mathrm{d}s)\\
&= \int_{s\neq 0}\frac{1-G(s)-sA}{s^2}(1+A)^{-2}\,\nu(\mathrm{d}s)
\end{aligned}$$

the integral being convergent in the operator norm. Putting together the pieces concludes the proof. $\square$

Remark 4.5. In the "unilateral" case, that is when one considers convolution semigroups instead of groups, the Lévy-Khintchine representation simplifies to

$$\psi(z) = c + dz + \int_{s>0} \left(1 - e^{-zs}\right) \phi(ds), \tag{7}$$

as shown by Phillips in [41]. By the same proof as above, using the functional calculus in the unilateral case, one recovers easily the unilateral analogue of Theorem 4.4 above:

(Unilateral Lévy-Khintchine formula [41]**)** *Let $S = (\mu_t)_{t\geq 0}$ be a positive convolution semigroup with log-characteristic function ψ given by the Lévy-Khintchine formula (7). Let $-A$ be the generator of a bounded C_0-semigroup on a Banach space X, and let $-A_S$ be the generator of the subordinate semigroup S_T given by*

$$S_T(t) := [e^{-t\psi}](A) = \int_{\mathbb{R}_+} T(s)\, \mu_t(\mathrm{d}s) \qquad (t \geq 0).$$

Then $\mathcal{D}(A) \subset \mathcal{D}(A_S)$ and

$$A_S x = cx + dAx + \int_0^\infty \Big(x - T(s)x\Big)\, \phi(ds), \;\; x \in \mathcal{D}(A).$$

4.2. **Fractional powers that are not odd integers.** Our next example generalises a result of Goldstein [21] for non-odd integer powers to all fractional powers that are not odd integers.

Theorem 4.6. *Let $-A$ be the generator of a bounded group. Then for all positive exponents α that are not odd integers, i.e. exponents $\alpha > 0$ for which there is an integer n with $2n-1 < \alpha < 2n+1$, the operator $-(-1)^n A^\alpha$ generates an analytic semigroup of angle $(1 - |2n - \alpha|)\pi/2$.*

Proof. For $0 < \alpha < 1$, $-A^\alpha$ generates an analytic semi-group of angle $(1-\alpha)\pi/2$ even in the case of $-A$ generating a bounded semigroup, see [9], [32, Thm. 5.4.1] or [24, Prop. 3.1.2]. Let $n \geq 1$ and $\psi(z) := (-1)^{n+1} z^\alpha$. We have to show that M_ψ generates a semi-group on $\mathrm{L}^1(\mathbb{R})$ and that this semigroup is analytic of angle $(1-|2n-\alpha|)\pi/2$, the rest is Corollary 4.2. We shall establish the equivalent resolvent estimate, i.e., we need to bound the $\mathcal{E}(i\mathbb{R})$-norm of $\lambda(\lambda-\psi)^{-1}$ for $\lambda = |\lambda| e^{i\theta}$ with

$$-(1 - |n - \alpha/2|)\pi + \varepsilon < \theta < (1 - |n - \alpha/2|)\pi - \varepsilon.$$

By Carlson's inequality (3) we obtain

$$\|R(\lambda, M_\psi)\|_{\mathcal{L}(X_0)} \;\leq\; C \left\| \frac{1}{\lambda - \psi} \right\|_{\mathrm{L}^2(i\mathbb{R})}^{1/2} \left\| \frac{\psi'}{(\lambda - \psi)^2} \right\|_{\mathrm{L}^2(i\mathbb{R})}^{1/2}.$$

Note that depending on the argument of $\psi(k)$, which is $\pm(1 - |n - \alpha/2|)\pi$,

$$|\lambda - (-1)^{n+1} k^\alpha|^2 = |\lambda|^2 + |k|^{2\alpha} - 2|\lambda||k|^\alpha \cos(\theta \pm (1 - |n - \alpha/2|)\pi).$$

Hence for $\alpha > 1/2$,

$$\begin{aligned}
\|R(\lambda, M_\psi)\|_{\mathcal{B}(X)} \leq & C \left(\int_{-\infty}^\infty \frac{1}{|\lambda|^2 + |k|^{2\alpha} - 2|\lambda||k|^\alpha \cos(\varepsilon)}\, dk \right)^{1/4} \\
& \left(\int_{-\infty}^\infty \frac{\alpha^2 |k|^{2\alpha-2}}{(|\lambda|^2 + |k|^{2\alpha} - 2|\lambda||k|^\alpha \cos(\varepsilon))^2}\, dk \right)^{1/4} \\
= & C \left(\int_0^\infty \frac{2|\lambda|^{-2+1/\alpha} u^{-1+1/\alpha}/\alpha}{1 + u^2 - 2u\cos(\varepsilon)}\, du \right)^{1/4} \\
& \left(\int_0^\infty \frac{2\alpha |(\lambda u)^{1/\alpha}|^{2\alpha-2} |\lambda|^{-4+1/\alpha} u^{-1+1/\alpha}}{(1 + u^2 - 2u\cos(\varepsilon))^2}\, du \right)^{1/4} \\
\leq & |\lambda|^{-1} \tilde{C}.
\end{aligned}$$

This is what we needed to prove. ☐

Remark 4.7. Theorem 4.6 is not surprising in view of the following heuristic argument. Since it is well known that $-A^2$ is sectorial of angle 0, also $(-1)^n A^{2n} = (-A^2)^n$ is sectorial of angle 0. On the other hand, supposing that A is injective, $A^{\alpha-2n}$ is sectorial of angle $|2n-\alpha|$ the product of these two commuting sectorial operators should then be sectorial of the appropriate angle. However, to the best of our knowledge, a statement precisely justifying the last step of this reasoning is missing in the literature. (Under some stronger assumptions it appears, e.g., in [27, Corollary 2.2].)

Theorem 4.6 also gives us an explicit generator formula for fractional powers.

Corollary 4.8. *Let $-A$ be the generator of a bounded C_0-group G, and let $\alpha > 0$ such there is $n \in \mathbb{N}$ with $2n-1 < \alpha < 2n+1$. Then*

$$(-1)^{n+1} A^\alpha x = \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{G(hs)x - x}{h^\alpha}\, \mu_1(\mathrm{d}s) \qquad (x \in \mathcal{D}(A^\alpha)), \tag{8}$$

where $(\mathcal{L}\mu_1)(z) = \exp((-1)^{n+1} z^\alpha)$, $z \in i\mathbb{R}$.

Proof. Take $\mu_t \in \mathrm{M}(\mathbb{R})$ such that $(\mathcal{L}\mu_t)(z) = e^{t(-1)^{n+1} z^\alpha} = e^{(-1)^{n+1}(t^{1/\alpha} z)^\alpha}$. Hence $\mathrm{d}\mu_t(s) = \mathrm{d}\mu_1(s/t^{1/\alpha})$. Furthermore, $\int \mu_1(\mathrm{d}u) = (\mathcal{L}\mu_1)(0) = 1$. Hence for $x \in \mathcal{D}(A^\alpha)$,

$$\begin{aligned}(-1)^{n+1} A^\alpha x &= \lim_{h\to 0^+} \frac{1}{h}\left(\int_{\mathbb{R}} G(s)x\, \mu_h(\mathrm{d}s) - x\right)\\ &= \lim_{h\to 0^+} \frac{1}{h}\int_{\mathbb{R}} \left(G(sh^{1/\alpha})x - x\right) \mu_1(\mathrm{d}s) = \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{G(hs)x - x}{h^\alpha}\, \mu_1(\mathrm{d}s).\end{aligned}$$

☐

Let $\mathrm{C}_0(\mathbb{R})$ denote the space of continuous functions vanishing at infinity. By taking G to be the shift semigroup on $\mathrm{C}_0(\mathbb{R})$ above, with $0 < \alpha \le 2$, formula (8) is a special case of the fractional derivative formulae obtained by Meerschaert and Scheffler in [36].

4.3. **Approximation formulae.** Another application is the transference of all approximation formulae in the $\mathrm{L}^1(\mathbb{R})$-setting (see, for example, [36]) of fractional derivatives to fractional powers. As an example we prove a shifted finite difference formula for the fractional powers based on a shifted Grünwald formula which was developed in [38] for the fractional derivatives of functions. A fractional difference formula was obtained by Westphal in [50] for generators of semigroups which was then used to obtain the one in the $\mathrm{L}^1(\mathbb{R}_+)$ setting. Following the philosophy of the present paper we show once again that it is easily done in the other direction, too: the $\mathrm{L}^1(\mathbb{R})$-result implies the abstract one. We also mention that the shift in the finite difference formula is not merely a useless generalisation as the conventional one produces nonstable approximations in fractional order advection-dispersion equations even when combined with the most robust time integration method, the Backward Euler method. The shifted ones remedy this problem as shown in [38].

Proposition 4.9. *Let $-A$ be the generator of a bounded C_0-group G on a Banach space X. Let $\alpha > 0$ and $p \in \mathbb{R}$. Then one has*

$$A^\alpha x = \lim_{h\to 0+} \frac{1}{h^\alpha} \sum_{n=0}^\infty (-1)^n \binom{\alpha}{n} G((n-p)h)x \qquad (x \in \mathcal{D}(A^\alpha)). \tag{9}$$

Proof. We fix $\alpha > 0$ and $p \in \mathbb{R}$. Note that since $\sum_n \left|\binom{\alpha}{n}\right| < \infty$ the sum converges absolutely in operator norm. Define $\phi(z) = z^\alpha$ and

$$\begin{aligned}\phi_h(z) &:= \frac{1}{h^\alpha}\sum_{n\geq 0}(-1)^n\binom{\alpha}{n}e^{-(n-p)hz} = \frac{e^{phz}}{h^\alpha}\sum_{n\geq 0}(-1)^n\binom{\alpha}{n}e^{-nhz}\\ &= \frac{e^{phz}}{h^\alpha}\left(1-e^{-hz}\right)^\alpha = z^\alpha e^{phz}\left(\frac{1-e^{-hz}}{hz}\right)^\alpha = z^\alpha e^{phz} f(hz)\end{aligned}$$

where $f(z) = ((1-e^{-z})/z)^\alpha$ is the Laplace–Stieltjes transform of a fractional spline as developed by Unser and Blu in [49]. Functional calculus yields

$$\phi_h(A) = A^\alpha G(-ph)f(hA) \supset G(-ph)f(hA)A^\alpha.$$

As $f \in \mathcal{E}(i\mathbb{R})$, Corollary 3.3 shows that $f(hA) \to I$ strongly, which implies what we want. □

5. Application to Fractional Advection-Dispersion Equations

In this section we are proposing a new model for solute transport in the subsurface based on subordinating a flow against a subordinator that is not unilateral; i.e. that has support on all of $\mathbb{R}$. Theorem 4.1 allows us to identify the governing partial differential equation which in turn enables us to extend the model. The Grunwald approximation formula, Proposition 4.9 provides us with a tool to numerically approximate the solution to the extended model. We first explain the idea omitting some technicalities and then in Theorem 5.1 we give the precise formulation of the equation and prove well-posedness within the framework the paper.

Fractional differential equations have recently made a renaissance, mainly driven by scientists in Physics [5, 10, 13, 31, 33, 35, 37, 45, 51], Finance [22, 42, 44, 46], and Hydrology [3, 6, 7, 8, 47, 48], as they can be derived via stochastic limit theorems and hence provide robust and parsimonious models predicting power-law tails. This is because fractional derivatives derive from sums of random movements with power law probability tails [47, 39], for which the usual central limit theorem is replaced by its heavy tail analogue [20, 34].

For example, for $1 < \alpha \leq 2$, the fractional advection-dispersion equation

$$\frac{\partial}{\partial t}u(t,x) = -v\frac{\partial}{\partial x}u(t,x) + D\left(\frac{\partial}{\partial x}\right)^\alpha u(t,x); \quad u(0,x) = f(x) \tag{10}$$

has been successfully applied in [8] to model solute transport in subsurface flow on a laboratory and field scale; i.e., a tracer was injected into the subsurface (and a sand box) and the concentration of tracer was measured at various points downstream. The data clearly shows non-Fickian dispersion and power-law leading tails, both characteristics captured by (10) as the Green's function solution to (10) is the density of a completely positively skewed stable random variable; i.e. the solution to (10) has Fourier transform

$$\hat{u}(t,k) = e^{t(-vik + D(ik)^\alpha)}\hat{f}(k).$$

A competing model was proposed by Baeumer et al. in [3] where the solution to

$$\frac{\partial}{\partial t}u(t,x) = -\left(v\frac{\partial}{\partial x} - D\frac{\partial^2}{\partial x^2}\right)^\beta u(t,x); \quad u(0,x) = f(x) \tag{11}$$

with $\beta = \alpha/2, \beta \leq 1$ was investigated. The solution is given through Bochner subordination of the solution $T(t)f$ to the classical advection dispersion equation ($\beta = 1$) against a β-stable density. In other words the solution is given by randomising the individual clock for different particles according to a β-stable random variable,

$$u(t,x) = \int_0^\infty g_\beta(t,s)T(s)f\,ds,$$

where the Laplace transform of g_β is given by $\int_0^\infty e^{-\lambda s} g_\beta(t,s)\,ds = e^{-t\lambda^\beta}$. This model is of course easily extended to higher dimensions and more complex flow fields as the concept of randomising time is independent of any given geometry. The main drawback is that the solution disperses too fast for many applications as it is basically a damped β stable random variable and as such has a divergent mean; the Green's function solution decays like $x^{-\alpha/2-1}$ for large x. Note that the subordinator had to be restricted to $\beta \le 1$ in order to keep it unilateral which is necessary for subordinating a semigroup that is not a group. Also note that $-\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} \neq \frac{\partial^\alpha}{\partial x^\alpha}$, as can be seen by their Fourier symbols $-|k|^\alpha \neq (ik)^\alpha$, which rules out obtaining a solution to (10) through classical unilateral subordination.

The techniques in this article however give us a tool to combine the two models and interpret the solutions to (10) as particles having randomised clocks (or random velocities) as for groups the subordinator no longer has to be unilateral and we can therefore assign negative local times (or negative velocities) to obtain asymmetric dispersion. Furthermore the subordinator can now be a stable density of index $1 < \alpha \le 2$, ensuring finite mean and the slightly faster decay of the Green's function solution for large x that was observed in the experiments. Knowing that the solution is given through a (non-unilateral) subordination we can then extend (10) to more complex geometries.

Observe that the solution $S(t)f$ to the one-dimensional fractional advection dispersion model (10) satisfies

$$S(t)f(x) = \int f(x-s)\mu^t(ds) = \int f(x-vs)\mu^t(vds) = \int [G(s)f](x)\mu^t(vds),$$

where G is the shift group generated by $-v\partial/\partial x$. Hence S can indeed be interpreted as being obtained by subordinating the one dimensional average flow (the 1-d flow group) against $\mu_S^t = \mu^t/v$ with Fourier transform

$$\widehat{\mu_S^t}(k) = e^{t(-ik+\gamma(ik)^\alpha)} \qquad (k \in \mathbb{R})$$

and $\gamma = Dv^{-1/\alpha}$ being a dispersion coefficient describing the spread of individual time around clock time.

Instead of subordinating the one-dimensional flow we now subordinate regional flows. A regional flow field $\{\vec{v}(x)\}$ can easily be extended to $\mathbb{R}^2$ such that the solution operator family $\{G(t)\}_{t\ge 0}$ of

$$\frac{\partial}{\partial t}u(t,x) = -\nabla_{\vec{v}}\, u(t,x); \qquad u(0,x) = f(x) \tag{12}$$

is a group on $\mathrm{C}_0(\mathbb{R}^2)$, where

$$\nabla_{\vec{v}}\, u(t,x) := \nabla\cdot(\vec{v}(x)u(x,t))\,.$$

Subordinating this group against μ_S^t; i.e, let

$$G_S(t)f = \int_{-\infty}^{\infty} G_S(s)f\mu_S^t(ds)$$

we obtain a semigroup $\{G_S(t)f\}_{t\ge 0}$ that is exhibiting fractional advection-dispersion along flow-lines. By Theorem 4.1, for $1 < \alpha \le 2$, it is the solution family to

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= -\nabla_{\vec{v}}\, u(t,x) + \gamma\left(\nabla_{\vec{v}}\right)^\alpha u(t,x)\\ u(0,x) &= f(x).\end{aligned} \tag{13}$$

One could, of course, obtain the same result by first transforming the flow geometry into a constant velocity rectangle by solving for the characteristics, then solving a constant-coefficient fractional-advection dispersion equation and finally transforming the solution back. The formulation in (13) however allows us to easily formulate and include a term modeling lateral dispersion, or concentration gradient driven diffusion. For example, allowing for concentration gradient driven diffusion yields the following Cauchy problem,

$$
\begin{aligned}
\frac{\partial}{\partial t}u(t,x) =& - \nabla_{\vec{v}}\, u(t,x) + \gamma \left(\nabla_{\vec{v}}\right)^{\alpha} u(t,x) + \sigma^2 \Delta u(t,x) \\
u(0,x) =& f(x).
\end{aligned}
\tag{14}
$$

Using the theory developed in the paper, the following theorem shows that the corresponding abstract Cauchy problem is well posed.

Theorem 5.1. *Assume that the vector field $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is twice continuously differentiable with uniformly bounded partial derivatives on $\mathbb{R}^2$ up to order 2. Let $X = \mathrm{C}_0(\mathbb{R}^2)$, $A = \sigma^2\Delta$ with maximal domain $\mathcal{D}(A) = \{f \in \mathrm{C}_0(\mathbb{R}) : \Delta f \in \mathrm{C}_0(\mathbb{R}^2)\}$*[1]*, $(B_0 f)(x) = \nabla\cdot(\vec{v}(x)f(x))$ with $\mathcal{D}(B_0) := \mathrm{C}^1_{\mathrm{c}}(\mathbb{R}^2)$ and $B := \overline{B_0}$. If σ is large enough, then the operator $\mathcal{A} := -B + \gamma B^\alpha + A$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A)$ generates a contractive analytic semigroup on X.*

Proof. It is well known that $-B$ generates a bounded C_0-group on X under the assumptions on $\vec{v}$ (see, for example, [19, Section II.3.29]). Since $\psi(z) := z - \gamma z^\alpha$ is a special case of the Lévy- Kintchine formula (6), we have that the multiplier $-\mathcal{M}_\psi$ generates a contraction semigroup on $L_1(\mathbb{R}^2)$ with measure having Fourier transform $e^{-t(z-\gamma z^\alpha)}$. By Theorem 4.1 the operator $-(z-\gamma z^\alpha)(B) = (-z+\gamma z^\alpha)(B)$ is the generator of the corresponding subordinated semigroup. It follows from Proposition 3.8 (c) that $(-z+\gamma z^\alpha)(B) = -B+\gamma B^\alpha$ and it is, in particular, closed. By Theorem 4.4, $\mathcal{D}(B^2) \subset \mathcal{D}(-B+\gamma B^\alpha)$ and since $\mathcal{D}(A) \subset \mathcal{D}(B^2)$ also $\mathcal{D}(A) \subset \mathcal{D}(-B+\gamma B^\alpha)$. Therefore, $-B+\gamma B^\alpha$ is A-bounded as A has a nonempty resolvent set. Moreover, A generates an analytic semigroup on X (see, for example, [1, Example 3.7.6]) and hence a standard perturbation result (see, for example [40, Chapter 3, Theorem 2.1]) shows that $\mathcal{A} = -B+\gamma B^\alpha + A$ generates an analytic semigroup on X if σ is large enough. Since both semigroups generated by $-B+\gamma B^\alpha$ and A are contractions, $\mathcal{A}$ generates a semigroup of contractions by the Trotter product formula. □

5.1. **Numerical Example.** As a numerical example we generated a divergence free flow field by putting a potential on an (approximately infinite) rectangle containing 2 ellipsoid obstacles. We then computed points on flowlines in intervals of Δt by solving the characteristic ODEs (Δt can vary from flowline to flowline). These points are useful for the purpose of solving (14) by an operator split method as we can use these points alternatively as mesh points for a finite element approximation of the concentration driven diffusion part as well as points used in the Grünwald approximation (see Proposition 4.9 which is the critical approximation result allowing us to approximate $-\nabla_{\vec{v}}\, u_n(t,x) + \tilde{D}\left(\nabla_{\vec{v}}\right)^\alpha u_n(t,x)$). The so generated mesh and flow lines can be seen in Figure 1. The resulting numerical solutions can be seen in Figures 2 and 3. Notice the leading tail in log-space in Figure 3 and the similarity of the main bulk of the respective plumes. However, the leading tail makes a big difference if, for example, the model is used to predict the evolution of a potentially toxic plume in groundwater, where even low concentrations are of significant interest.

References

1. W. Arendt, Ch. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics **96**, Birkhäuser Verlag, 2001.
2. B. Baeumer, M. Kovács, M. M. Meerschaert, Subordinated multiparameter groups of linear operators: Properties via the transference principle, in: H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise (Eds.), Functional Analysis and Evolution Equations, Dedicated to Gunter Lumer, Birkhäuser Verlag, 2008, pp. 35–50.

[1] Here we identify $\mathrm{C}_0(\mathbb{R}^2)$ with a subspace of distributions.

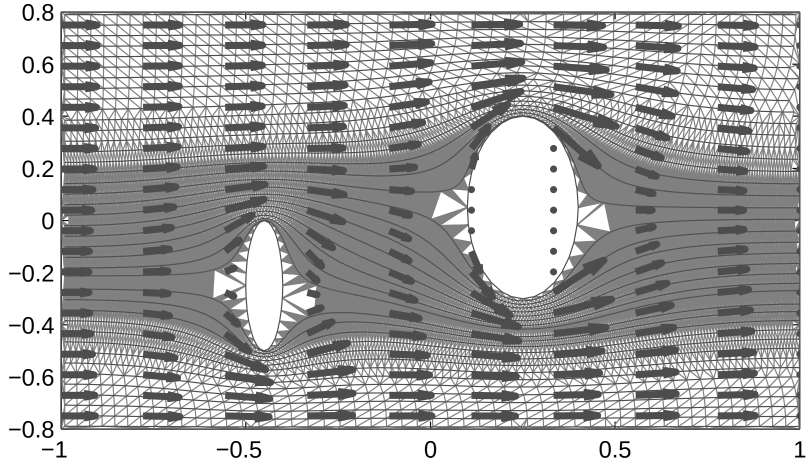

FIGURE 1. Flow field and mesh of numerical example.

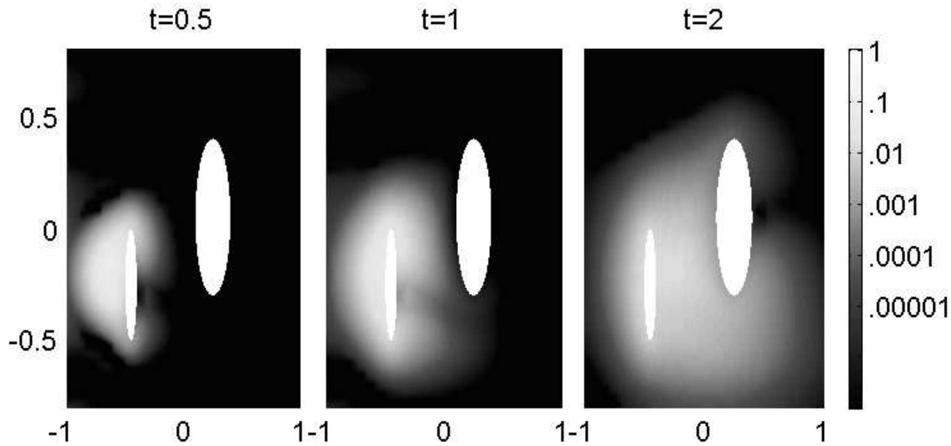

FIGURE 2. Snapshots of a numerical approximation of solutions to (14) with $\alpha = 2, \tilde{D} = .05, \sigma^2 = .005$ and initial condition $f(x) = \exp(-\frac{(x+.75)^2+(y+.2)^2}{.002})/2$.

3. B. Baeumer, M. M. Meerschaert, D. A. Benson and S. W. Wheatcraft, Subordinated advection-dispersion equation for contaminant transport, *Water Resources Research* **37** (2001), 1543–1550.
4. A. V. Balakrishnan, An operational calculus for infinitesimal generators of semigroups, *Trans. Amer. Math. Soc.* **91** (1959), 330–353.
5. E. Barkai, R. Metzler and J. Klafter, From continuous time random walks to the fractional Fokker-Planck equation. *Phys. Rev. E* **61** (2000), 132–138.
6. D. Benson, S. Wheatcraft and M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resources Research* **36**(6) (2000), 1403–1412.
7. D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, The fractional-order governing equation of Lévy motion, *Water Resources Research* **36** (2000), 1413–1424.
8. D. A. Benson, R. Schumer, M. M. Meerschaert and S. W. Wheatcraft, Fractional dispersion, Lévy motion, and the MADE tracer tests, *Transport in Porous Media* **42**(1–2) (2001), 211–240.
9. C. Berg, K. Boyadzhiev and R. DeLaubenfels, Generation of generators of holomorphic semigroups, *J. Austral. Math. Soc. (Series A)* **55** (1993), 246–269.
10. A. Blumen, G. Zumofen and J. Klafter, Transport aspects in anomalous diffusion: Lévy walks, *Phys. Rev. A* **40** (1989), 3964–3973.

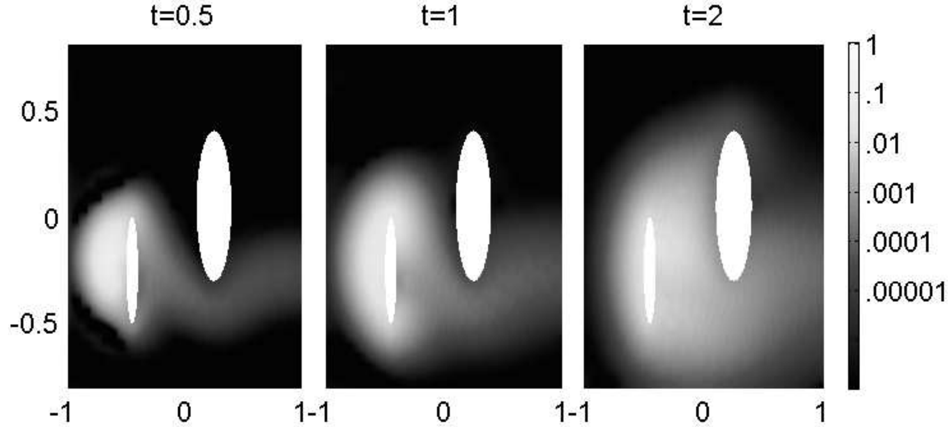

FIGURE 3. Snapshots of a numerical approximation of solutions to (14) with $\alpha = 1.3, \tilde{D} = -.05/\cos(\pi\alpha/2), \sigma^2 = .005$ and initial condition $f(x) = \exp(-\frac{(x+.75)^2+(y+.2)^2}{.002})/2$.

11. S. Bochner, Stochastic processes, *The Annals of Mathematics* **48** (4) (1947) 1014–1061.
12. S. Bochner, Diffusion equations and stochastic processes, *Proc. Natl. Acad. Sci. U. S. A.* **35** (7) (1949) 368–370.
13. J. Bouchaud and A. Georges, Anomalous diffusion in disordered media - statistical mechanisms, models and physical applications, *Phys. Rep.* **195** (1990), 127–293.
14. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Vol 1, Academic Press, 1971.
15. A. S. Carasso and T. Kato, On Subordinated holomorphic semigroups, *Trans. Amer. Math. Soc.* **327** (1991), 867–878.
16. F. Carlson, Une Inégalité, *Ark Mat.* **25B** (1935), 1–5.
17. S. Clark, *Sums of Operator Logarithms*, Quart. J. Math. 00 (2008), 1-15. Published Advance Access on 17 July 2008. doi:10.1093/qmath/han010.
18. R. R. Coifman and G. Weiss, *Transference methods in analysis*, Regional Conference Series in Mathematics **31**, American Mathematical Society, 1977.
19. K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.
20. W. Feller, *An Introduction to Probability Theory and Applications. Volumes I and II*, John Wiley and Sons, 1966.
21. J. A. Goldstein, Some remarks on infinitesimal generators of analytic semigroups, *Proc. Amer. Math. Soc.* **22** (1969), 91–93.
22. R. Gorenflo, F. Mainardi, E. Scalas and M. Raberto, Problems with using existing transport models to describe microbial transport in porous media, *Trends Math.*, American Society for Microbiology, Mathematical finance (Konstanz, 2000) (2001), 171–180.
23. M. Haase, A general framework for holomorphic functional calculi. *Proc. Edin. Math. Soc.* **48** (2005) 423–444.
24. M. Haase, *The Functional Calculus for Sectorial Operators.* Number 169 in Operator Theory: Advances and Applications. Birkhäuser-Verlag, Basel, 2006.
25. M. Haase, Functional calculus for groups and applications to evolution equations. *J. Evol. Equ.* **7** (2007), 529–554.
26. M. Haase, *On abstract functional calculus extensions.* To appear in: *Tübinger Berichte zur Funktionalanalysis* **16**, 2007/2008.
27. R. Haller-Dintelmann, M. Hieber, H^∞-calculus for the product of Non-Commuting operators, *Math. Z.* **251** (2005), 85–100.
28. M. Hieber and J. Prüss, Functional calculi for linear operators in vector-valued L^p-spaces via the transference principle, *Adv. Differential Equations* **3** (1998), 847–872.
29. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Colloquium Publications **31**, American Mathematical Society, 1957.
30. N. Jacob and R.L. Schilling, An analytic proof of the Lévy-Khinchin formula on $\mathbb{R}^n$, *Publ. Math. Debrecen* **53**(1–2) (1998), 69–89.
31. J. Klafter, A. Blumen and M. F. Shlesinger, Stochastic pathways to anomalous diffusion, *Phys. Rev. A* **35** (1987), 3081–3085.

32. C. Martinez Carracedo and M. Sanz Alix, *The Theory of Fractional Powers of Operators*, North-Holland Mathematics Studies **187**, Elsevier, 2001.
33. M. M. Meerschaert, D. A. Benson and B. Baeumer, Multidimensional advection and fractional dispersion, *Phys. Rev. E* **59** (1999), 5026–5028.
34. M. M. Meerschaert and H. P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, Wiley Interscience, New York, 2001.
35. M. M. Meerschaert, Benson, D.A. and B. Baeumer, Operator Lévy motion and multiscaling anomalous diffusion, *Phys. Rev. E* **63** (2001), 1112–1117.
36. M. M. Meerschaert and H.-P. Scheffler, Semistable Lévy motion. *Fractional Calculus and Applied Analysis*, **5**(1) (2002), 27–54.
37. M. M. Meerschaert, D. A. Benson, H. P. Scheffler and B. Baeumer, Stochastic solution of space-time fractional diffusion equations, *Phys. Rev. E* **65** (2002), 1103–1106.
38. M. M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advestion-dispersion equations, *J. Comput. Appl. Math.* **172** (2004), 65–77.
39. M. M. Meerschaert, J. Mortensen and S. Wheatcraft, Fractional vector calculus for fractional advection-dispersion, *Physica A* **367** (2006), 181–190.
40. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag New York, 1983.
41. R.S. Phillips, On the generation of semigroups of linear operators, *Pacific J. Math.* **2** (1952), 343–369.
42. M. Raberto, E. Scalas and F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, *Physica A*, **314** (2002), 749–755.
43. W. Rudin, *Functional Analysis. 2nd ed.*, International Series in Pure and Applied Mathematics. New York, NY: McGraw-Hill. xviii, 424 p. , 1991.
44. L. Sabatelli, S. Keating, J. Dudley and P. Richmond, Waiting time distributions in financial markets, *Eur. Phys. J. B* **27** (2002), 273–275.
45. A. I. Saichev and G. M. Zaslavsky, Fractional kinetic equations: solutions and applications, *Chaos* **7**(4) (1997), 753–764.
46. E. Scalas, R. Gorenflo and F. Mainardi, Fractional calculus and continuous-time finance, *Phys. A* **284** (2000), 376–384.
47. R. Schumer, D. A. Benson, M. M. Meerschaert and S. W. Wheatcraft, Eulerian derivation of the fractional advection-dispersion equation, *Journal of Contaminant Hydrology* **48** (2001), 69–88.
48. R. Schumer, D. A. Benson, M. M. Meerschaert and B. Baeumer, Multiscaling fractional advection-dispersion equations and their solutions, *Water Resources Research* **39** (2003), 1022–1032.
49. M. Unser and T. Blu, Fractional splines and wavelets, *SIAM Rev.* **42** (2000), 43–67.
50. U. Westphal, An approach to fractional powers of operators via fractional differences, *Proc. London Math. Soc.* **29** (1974), 557–576.
51. G. Zaslavsky, Fractional kinetic equation for hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion, *Phys. D* **76** (1994), 110–122.

B. Baeumer, Department of Mathematics, University of Otago, Dunedin, New Zealand, Ph: +64-3-479-7763, FAX: +64-3-479-8427
E-mail address: bbaeumer@maths.otago.ac.nz

M. Haase, Delft Institute of Applied Mathematics, Delft University of Technology, P.O.Box 5031, 2600 GA Delft, The Netherlands
E-mail address: m.h.a.haase@tudelft.nl

M. Kovács, Department of Mathematics, University of Otago, Dunedin, New Zealand, Ph: +64-3-479-7783, FAX: +64-3-479-8427
E-mail address: mkovacs@maths.otago.ac.nz