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Stochastic models for relativistic diffusion

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The diffusion equation is related to the Schrödinger equation by analytic continuation. The formula $E^2=p^2c^2+m^2c^4$ leads to a relativistic Schrödinger equation, and analytic continuation yields a relativistic diffusion equation that involves fractional calculus. This paper develops stochastic models for relativistic diffusion and equivalent differential equations with no fractional derivatives. Connections to anomalous diffusion are also discussed, along with alternative models.

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I. INTRODUCTION

Relativistic diffusion attempts to correct the traditional diffusion model for relativistic effects. Several different models have been proposed [1–3] (see also the recent review [4]). Here, we emphasize the connection to quantum mechanics. The Schrödinger equation reduces to the diffusion equation under analytic continuation. Using the total-energy formula $E^2=p^2c^2+m^2c^4$ from special relativity leads to a relativistic Schrödinger equation, and then analytic continuation yields a relativistic diffusion equation. This modified diffusion equation replaces the usual Laplacian $(\hbar^2/2m)\Delta$ with a fractional calculus analog $mc^2-\sqrt{m^2c^4-\hbar^2c^2\Delta}$ using the 1/2 power of the operator $m^2c^4-\hbar^2c^2\Delta$. This paper develops a stochastic model, based on the normal inverse Gaussian (NIG) distribution, for relativistic diffusion. An alternative differential equation is also derived, equivalent to the relativistic diffusion equation, but involving no fractional derivatives. Finally, some alternatives are discussed.

It is well known that the diffusion equation is related to the Schrödinger equation by analytic continuation [5]: use the wave function $\psi(x,t)=e^{i\mathbf{k}\cdot\mathbf{x}-it\omega}$, energy $E=\hbar\omega$ [6], and momentum $p=\hbar k$ [7] to compute

$$i\hbar\frac{\partial\psi}{\partial t}=(i\hbar)(-i\omega)\psi=\hbar\omega\psi=E\psi, \tag{1}$$

$$\hbar^2\Delta\psi=\hbar^2(ik)\cdot(ik)\psi=-(p\cdot p)\psi. \tag{2}$$

The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi \tag{3}$$

follows from Eqs. (1) and (2) using kinetic energy $E=p\cdot p/2m$. Analytic continuation $\tau=it$ yields the classical diffusion equation

$$\hbar\frac{\partial}{\partial\tau}\psi(x,\tau)=\frac{\hbar^2}{2m}\Delta\psi(x,\tau). \tag{4}$$

This justifies the formula $k=-i\nabla$ or $p=-i\hbar\nabla$ from pseudodifferential operator theory [8], since from Eq. (1)

$$i\hbar\frac{\partial\psi}{\partial t}=E\psi=\frac{p\cdot p}{2m}\psi=\frac{\hbar^2}{2m}(-i\nabla)\cdot(-i\nabla)\psi=-\frac{\hbar^2}{2m}\Delta\psi$$

using $\nabla\cdot\nabla=\Delta$. The diffusion equation (4) is usually written in the form

$$\frac{\partial}{\partial\tau}\psi(x,\tau)=D\Delta\psi(x,\tau), \tag{5}$$

which reduces to Eq. (4) if we set $D=\hbar/2m$. The point-source solution to Eq. (5) is a product of one-dimensional normal densities with zero mean and variance $2Dt$, so the diffusivity codes the spreading rate of a particle cloud. In classical mechanics, the diffusivity D depends on particle mass and other environmental parameters. In quantum mechanics, the diffusivity is inversely proportional to particle mass since larger particles scatter more slowly.

The total energy of a relativistic particle with rest mass m satisfies $E^2=p^2c^2+m^2c^4$, so the relativistic kinetic energy is $E=\sqrt{p^2c^2+m^2c^4}-mc^2$, which leads to a relativistic Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial\psi}{\partial t}&=E\psi=[\sqrt{p^2c^2+m^2c^4}-mc^2]\psi\\&=[\sqrt{m^2c^4-\hbar^2c^2\Delta}-mc^2]\psi.\end{aligned} \tag{6}$$

Note that, on one hand, in the low-mass limit this equation reduces to

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$$i\hbar\frac{\partial\psi}{\partial t}=(-\hbar^2c^2\Delta)^{1/2}\psi,$$

the Riesz fractional Schrödinger equation as proposed by [9]. On the other hand, in the nonrelativistic limit, $c\to\infty$, we recover the classical Schrödinger equation as

$$\lim_{c\to\infty}\sqrt{m^2c^4-\hbar^2c^2\Delta}-mc^2=\lim_{c\to\infty}\frac{\sqrt{m^2-\hbar^2\Delta/c^2}-m}{1/c^2}=\lim_{c\to\infty}\frac{\left(\dfrac{2\hbar^2\Delta/c^3}{2\sqrt{m^2-\hbar^2\Delta/c^2}}\right)}{-2/c^3}=-\frac{\hbar^2\Delta}{2m} \quad (7)$$

using L'Hôpital's rule.

Now in the same sprit as before, analytic continuation $\tau=it$ of Eq. (6) leads to the relativistic diffusion equation

$$\hbar\frac{\partial}{\partial\tau}\psi(x,\tau)=[mc^2-\sqrt{m^2c^4-\hbar^2c^2\Delta}]\psi(x,\tau). \quad (8)$$

Substituting $\bar{m}=mc^2/\hbar$ and $t=\tau$ we obtain

$$\frac{\partial}{\partial t}\psi(x,t)=[\bar{m}-\sqrt{\bar{m}^2-c^2\Delta}]\psi(x,t), \quad (9)$$

the form used in [1,10] with $c=1$.

II. RELATIVISTIC DIFFUSION

Transform methods applied to Eq. (9) yield useful stochastic models for relativistic diffusion. The Fourier transform (FT) is $\hat{f}(k)=\int e^{-ik\cdot x}f(x)dx$, so that $-\|k\|^2\hat{f}(k)$ is the FT of $\Delta f(x)$. Take FT in Eq. (9) to get $\hat{\psi}'(k,t)=[\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2}]\hat{\psi}(k,t)$, whose point-source solution,

$$\hat{\psi}(k,t)=\exp[t(\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2})], \quad (10)$$

is the FT of a NIG probability density function (pdf) [11]. The NIG process with pdf $\psi(x,t)$ is $A(E_t)$, where the outer process $A(t)$ is a Brownian motion and $E_t=\inf\{\tau>0:B(\tau)+\bar{m}\tau>t\}$ is the hitting time of another independent Brownian motion with drift. The NIG model has been used in finance and turbulence, as an alternative to Brownian motion, with heavier tails [12,13]. We have shown that the NIG process traces the path of a particle undergoing relativistic diffusion.

A Taylor expansion $\sqrt{\bar{m}^2+x}-\bar{m}\approx ax$ with $a=1/2\bar{m}$ shows that

$$\hat{\psi}(k,t)\approx\exp\left(-\frac{tc^2}{2\bar{m}}\|k\|^2\right)=\exp\left(-\frac{t\hbar}{2m}\|k\|^2\right), \quad (11)$$

so at late time or large $\bar{m}$ the relativistic diffusion process $A(E_t)\approx A(t)$, a traditional diffusion process. In the low-mass limit $\bar{m}\to0$ of Eq. (10) we have

$$\hat{\psi}(k,t)\approx\exp(-tc\|k\|),$$

so that $A(E_t)\approx C(t)$ a Cauchy process whose pdf $q(x,t)$ solves the anomalous superdiffusion equation $\partial q/\partial t=-(-c^2\Delta)^{1/2}q$ as in [14]. The superdiffusion leads to fast particle spreading, which is facilitated by a smaller particle mass. Figure 1 illustrates the transition from Cauchy at early time to Gaussian at late time, for the NIG process. This is a type of transient anomalous superdiffusion [15]. Figure 2 shows the transition from Gaussian to Cauchy as mass decreases.

The NIG solution to the relativistic diffusion equation (9) follows the scaling relationship

$$\psi(x,t,\bar{m})=\lambda\psi(\lambda x,\lambda t,\bar{m}/\lambda), \quad (12)$$

since $\hat{\psi}(k/\lambda,\lambda t,\bar{m}/\lambda)=\hat{\psi}(k,t,\bar{m})$. Here, we use the fact that $\hat{f}(k/\lambda)$ is the FT of $\lambda f(\lambda x)$ in general. The scaling relation (12) can be used along with Figs. 1 and 2 to visualize solutions to the relativistic diffusion equation (9) in any application. Also note that the variance of the NIG solution is given by [16]

$$\sigma_\psi^2(t)=\frac{c^2}{\bar{m}}t=\frac{\hbar}{m}t. \quad (13)$$

This is identical to the variance of the solution to the classical diffusion equation (5). This implies a narrower sharper

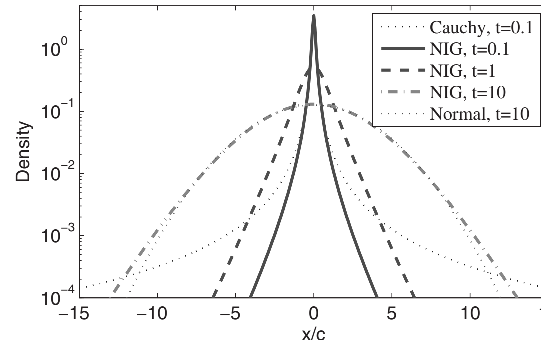

FIG. 1. Point-source solution to the relativistic diffusion equation (9) with $\bar{m}=1$, compared with Cauchy approximation for small t and Gaussian approximation for large t. Note the scaling relation (12).

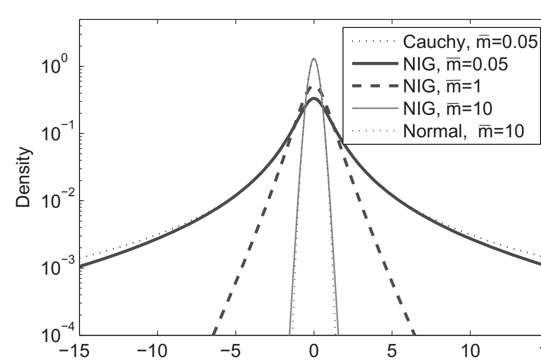

FIG. 2. Point-source solution to the relativistic diffusion equation (9) with $t=1$, compared with Cauchy approximation for small $\bar{m}$ and Gaussian approximation for large $\bar{m}$. Note the scaling relation (12).

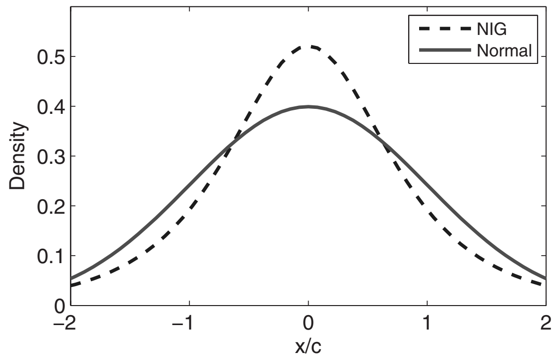

FIG. 3. The point-source solution to the relativistic diffusion equation (9) for $\bar{m}=1$ and $t=1$ is narrower, with a sharper peak, than the corresponding solution to the traditional diffusion equation (5). Both have the same variance (13).

peak but heavier tail for early time or low masses as seen in Fig. 3.

III. CONTINUOUS TIME RANDOM WALK DERIVATION OF THE NIG MODEL

The NIG model for relativistic diffusion can be better understood in terms of an equivalent differential equation. Write the Laplace transform (LT) $\tilde{h}(s)=\int e^{-st}h(t)dt$, so that $s\tilde{h}(s)-h(0)$ is the LT of $h'(t)$, and let $\bar{\psi}(k,s)=\int e^{-st}\hat{\psi}(k,t)dt$ denote the Fourier-Laplace transform (FLT). The FLT of Eq. (9) is $s\bar{\psi}(k,s)-\hat{\psi}_0(k)=[\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2}]\bar{\psi}(k,s)$. Solve to get

$$\bar{\psi}(k,s)=\frac{\hat{\psi}_0(k)}{s-\bar{m}+\sqrt{\bar{m}^2+c^2\|k\|^2}}. \tag{14}$$

Some algebraic manipulation yields

$$\bar{\psi}(k,s)=\frac{\hat{\psi}_0(k)}{s-\bar{m}+\sqrt{\bar{m}^2+c^2\|k\|^2}}\frac{s-\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2}}{s-\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2}}$$
$$=\frac{[s-\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2}]\hat{\psi}_0(k)}{s^2-2\bar{m}s-c^2\|k\|^2}.$$

Rearrange to get

$$-[s^2\bar{\psi}(k,s)-s\hat{\psi}_0(k)-(\bar{m}-\sqrt{\bar{m}^2+c^2\|k\|^2})\hat{\psi}_0(k)]$$
$$+2\bar{m}[s\bar{\psi}(k,s)-\hat{\psi}_0(k)]=-c^2\|k\|^2\bar{\psi}(k,s),$$

divide by $2\bar{m}$ and invert the FLT to arrive at

$$-\frac{1}{2\bar{m}}\frac{\partial^2}{\partial t^2}\psi(x,t)+\frac{\partial}{\partial t}\psi(x,t)=\frac{c^2}{2\bar{m}}\Delta\psi(x,t), \tag{15}$$

with initial conditions $\psi(x,0)=\psi_0(x)$ and $\psi'(x,0)=[\bar{m}-\sqrt{\bar{m}^2-c^2\Delta}]\psi_0(x)$. Equation (15) is mathematically equivalent to the relativistic diffusion equation, but contains no fractional calculus operators. It is an elliptic equation in space-time and the condition for $\psi'(x,0)$ is equivalent to the boundary condition (in space-time) that $\lim_{t\to\infty}\psi(t,x)=0$.

The differential equation (15) governs the limit density of a continuous time random walk (CTRW) [17]: the sum $S(n)=X_1+\cdots+X_n$ gives the particle location after n jumps. For identically distributed jumps with zero mean and finite variance, as the time scale $r\to\infty$, the central limit theorem yields $r^{-1/2}S(rt)\Rightarrow A(t)$, a Brownian motion whose probability densities $p(x,t)$ solve the diffusion equation (5); i.e., $\partial p/\partial t=(c^2/2\bar{m})\Delta p$ which has the propagator

$$p(x,t)=\exp\left(-\frac{x^2}{2c^2t/\bar{m}}\right)/\sqrt{2\pi c^2t/\bar{m}}$$

as a solution.

In a CTRW, a random waiting time $J_n>0$ precedes the nth jump, which therefore occurs at time $T(n)=J_1+\cdots+J_n$. The number of jumps by time t is $N_t=\max\{n:T(n)\le t\}$; hence, the particle location at time t is $S(N_t)$. Normalize so that $\langle J_n\rangle=1$. Then $N_t\approx t$, and $S(N_t)\approx A(t)$ as $t\to\infty$.

To illuminate the second-order behavior of the solution to Eq. (15), apply a two-scale limiting procedure, segregating the mean waiting time, and the deviation from the mean. Now $r^{-1/2}[T(rt)-rt]+r^{-1}[rt]\Rightarrow B(t)+t$, an independent Brownian motion with drift whose density $f(x,t)$ solves $\partial f/\partial t=-\partial f/\partial x+\frac{1}{2\bar{m}}\partial^2 f/\partial x^2$. The inverse process N_t converges to the inverse limit $E_t=\inf\{\tau>0:B(\tau)+\tau>t\}$; hence, the CTRW $S(N_t)\approx A(E_t)$. The CTRW limit $A(E_t)$ has NIG density

$$\psi(x,t)=\int_0^\infty p(x,\tau)h(\tau,t)d\tau, \tag{16}$$

where

$$h(\tau,t)=(t/\tau)\exp\left(-\frac{(t-\tau)^2}{2\tau/\bar{m}}\right)/\sqrt{2\pi\tau/\bar{m}} \tag{17}$$

is the inverse Gaussian pdf of E_t [17]. A Laplace transform calculation [17] shows that $\psi(x,t)$ solves the differential equation (15), which can be considered as a relativistic correction to the diffusion equation (4) to account for random waiting times between particle jumps. The density of $h(\tau,1)$ is displayed in Fig. 4. The subordination formula (16) was used to plot the NIG density in Figs. 1–3.

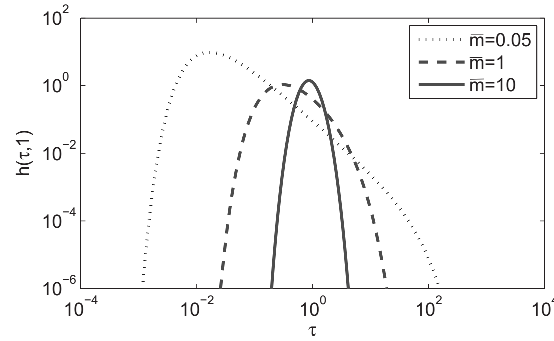

FIG. 4. The operational time density $h(\tau,1)$ from Eq. (17) accounts for random waiting times between particle motions.

IV. MODEL EXTENSIONS AND ALTERNATIVES

An equivalent stochastic model for relativistic diffusion uses tempered stable laws [18]. The NIG relativistic diffusion process can also be written in the form $B(T_t)$, where $B(t)$ is a Brownian motion and T_t is a tempered stable process [10]. The tempered stable process modifies the stable subordinator from the theory of anomalous diffusion, by exponentially tempering its power-law jumps. The pdf of T_t has LT $\tilde{h}(s,t)=\exp\{t[\lambda^{\beta}-(\lambda+s)^{\beta}]\}$. Take $\lambda=\bar{m}^2$ and $\beta=1/2$, and suppose that the pdf of $B(t)$ has FT $\hat{p}(k,t)=e^{-t\|ck\|^2}$. Then the pdf of $B(T_t)$ has FT

$$\hat{\psi}(k,t) = \int_0^{\infty} e^{-\tau\|k\|^2} h(\tau,t)d\tau = e^{t[\bar{m}-\sqrt{\bar{m}^2+\|ck\|^2}]}$$

using the LT formula for h, so that ψ solves the relativistic diffusion equation (9).

Now the relativistic diffusion process $B(T_t)$ can be seen as a transient anomalous diffusion. Anomalous superdiffusion operates by speeding up time in the diffusion process, due to forward jumps in the inner process, with power-law statistics. Relativistic diffusion moderates those forward jumps by an exponential cooling, at a rate that increases as the square of the particle mass, and this eventually pulls the pdf back into a Gaussian shape at late time.

Next we consider an extension to the relativistic diffusion model. The relativistic stable process [10] $B(T_t)$ has a pdf with FT $\hat{\psi}(k,t)=\exp\{t[\lambda^{\beta}-(\lambda+\|k\|^2)^{\beta}]\}$ for $0<\beta<1$. It reduces to the relativistic diffusion process when $\lambda=m^2$ and $\beta=1/2$. The pdf solves a pseudodifferential equation

$$\frac{\partial}{\partial t}\psi(x,t) = [\lambda^{\beta} - (\lambda - \Delta)^{\beta}]\psi(x,t) \tag{18}$$

that extends Eq. (9). The low-mass limit $\hat{\psi}(k,t) \approx \exp(-t\|k\|^{2\beta})$ is a symmetric stable process with index $\alpha=2\beta$ that solves the fractional diffusion equation $\partial\psi/\partial t=-(-\Delta)^{\beta}\psi$ in [14]. Equation (18) is an alternative to the fractional diffusion equation with nicer spectral properties [1,10].

An alternative path to relativistic diffusion was suggested by [3] and is connected to the telegrapher's equation

$$\frac{\partial^2}{\partial t^2}\psi + 2a\frac{\partial}{\partial t}\psi = v^2\Delta\psi. \tag{19}$$

The underlying stochastic model is a process where particles travel at a constant velocity v and change direction after a Poisson waiting time with rate a. Formally solving for the time-differential operator leads to a different relativistic diffusion equation which is not equivalent to Eq. (9),

$$\frac{\partial}{\partial t}\psi = [-a + \sqrt{a^2 + v^2\Delta}]\psi. \tag{20}$$

To derive Eq. (20) from the relativistic Schrödinger equation, use a different analytic continuation, as suggested by [3]: replace $\hbar$ with $-i\hbar$ in Eq. (6) to get

$$\hbar\frac{\partial\psi}{\partial t} = [-mc^2 + \sqrt{m^2c^4 + \hbar^2c^2\Delta}]\psi, \tag{21}$$

and then choose $v=c$ and $a=\bar{m}=mc^2/\hbar$ to arrive at Eq. (20). However, the solutions to Eq. (20) are not probability densities, or even real valued, since for any real-valued function $\hat{f}(k)$ is the complex conjugate of $\hat{f}(-k)$ and this is not the case for

$$\hat{\psi}(k,t) = \exp[t(-\bar{m} + \sqrt{\bar{m}^2 - c^2\|k\|^2})]$$

when $\|k\|>\bar{m}/c$, making a physical interpretation of Eq. (20) rather difficult.

V. DISCUSSION

Some literature on relativistic diffusion [2] has suggested that even the traditional diffusion equation requires correction, because the propagator $p(x,t)=\exp(-x^2/4Dt)/\sqrt{4\pi Dt}$ assigns some small probability mass to locations $|x|>ct$ which cannot be reached by a single particle without exceeding the speed of light c. The resulting stochastic process is a Gaussian diffusion with the superluminal locations excluded. The process avoids the sharp fronts in the usual telegrapher's equation (19), but is also non-Markovian.

An alternative view, which is completely physical, is that the equation $\partial p/\partial t=D\Delta p$ governs the long-time asymptotic limit of a random walk with physical bounds on velocity. The CTRW model is suited for this purpose [19]. CTRW particle motions are subluminal so long as $\|X_n\|<cJ_n$ in general. For example, we can place a lower bound $J_n>b$ on the waiting times between jumps and an upper bound $\|X_n\|\leq B$ on the jumps, where $B/b<c$. Another option is to employ a coupled CTRW as in [19], where the distribution of the jumps length X_n depends on the previous waiting time J_n, so that all motions stay inside the light cone. This was the original motivation for the coupled CTRW model.

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