REFLECTED SPECTRALLY NEGATIVE STABLE PROCESSES AND THEIR GOVERNING EQUATIONS

BORIS BAEUMER, MIHÁLY KOVÁCS, MARK M. MEERSCHAERT, RENÉ L. SCHILLING, AND PETER STRAKA

Abstract. This paper explicitly computes the transition densities of a spectrally negative stable process with index greater than one, reflected at its infimum. First we derive the forward equation using the theory of sun-dual semigroups. The resulting forward equation is a boundary value problem on the positive half-line that involves a negative Riemann-Liouville fractional derivative in space, and a fractional reflecting boundary condition at the origin. Then we apply numerical methods to explicitly compute the transition density of this space-inhomogeneous Markov process, for any starting point, to any desired degree of accuracy. Finally, we discuss an application to fractional Cauchy problems, which involve a positive Caputo fractional derivative in time.

1. Introduction

Consider a spectrally negative (no positive jumps) stable Lévy process Y_t with characteristic function

$$\mathbb{E}[e^{ikY_t}] = e^{t(ik)^\alpha} \tag{1.1}$$

for some $1 < \alpha \leqslant 2$. If $\alpha = 2$, then Y_t is a Brownian motion with variance $2t$. Now define

$$Z_t = Y_t - \inf\{Y_s : 0 \leqslant s \leqslant t\}. \tag{1.2}$$

The reflected stable process (1.2) is also the recurrent extension of the process Y_t killed at zero, which instantaneously and continuously leaves zero, see Patie and Simon [42]. Let $C_\infty(\mathbb{R})$ denote the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ that tend to zero as $|x| \to \infty$, with the supremum norm. We say that a time-homogeneous Markov process X_t is a *Feller process* if the semigroup $T_t f(x) = \mathbb{E}[f(X_{t+s})|X_s = x]$ satisfies $T_t f \in C_\infty(\mathbb{R})$ and $T_t f \to f$ as $t \to 0$ in the Banach space (supremum) norm, for all $f \in C_\infty(\mathbb{R})$. It is not hard to show (see Theorem 2.1 in this paper) that Z_t is a Feller process, and since $Z_t \geqslant 0$ by definition, the space-inhomogeneous Markov process Z_t lives on the state space $[0, \infty)$.

If $\alpha = 2$, then this process is called "reflected Brownian motion" and the governing differential equation (forward Kolmogorov equation) for the transition density $p(x, y, t)$

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of $Z_{t+s} = y$ given $Z_s = x$ is the diffusion equation $\partial_t p(x,y,t) = \partial_y^2 p(x,y,t)$ together with the reflecting boundary condition

$$\partial_y p(x,y,t)\Big|_{y=0+} := \lim_{h\to 0+} \frac{p(x,y+h,t)-p(x,y,t)}{h}\Big|_{y=0} = 0 \quad \text{for all } t>0, \tag{1.3}$$

(i.e., the normal derivative vanishes) see for example Itô and McKean [25, Eq. 8]. This paper extends to the case of a reflected stable process. The stable process Y_t without reflection is a space-homogeneous Markov process, so its transition density $p(y,t)$ is independent of the initial state x. This density solves a fractional diffusion equation, $\partial_t p(y,t) = D^{\alpha}_{-y} p(y,t)$ that involves a negative Riemann-Liouville fractional derivative in space, see (2.4) below for the definition. The fractional derivative reduces to the usual second derivative in the case $\alpha = 2$. The forward equation for the reflected stable process turns out to be the fractional diffusion equation $\partial_t p(x,y,t) = D^{\alpha}_{-y} p(x,y,t)$ with the fractional reflecting boundary condition

$$D^{\alpha-1}_{-y} p(x,y,t)\Big|_{y=0+} := \lim_{h\to 0+} \frac{1}{h^{\alpha-1}} \sum_{k=0}^{\infty} w_k^{\alpha-1} p(x,y+kh,t)\Big|_{y=0} = 0 \quad \text{for all } t>0, \tag{1.4}$$

using the (fractional) binomial coefficients

$$w_k^{\alpha} := (-1)^k \binom{\alpha}{k}.$$

When $\alpha = 2$ we have $w_0^{\alpha-1} = 1$, $w_1^{\alpha-1} = -1$, and $w_k^{\alpha-1} = 0$ for $k > 1$, so that (1.4) reduces to the classical condition (1.3), i.e., the one-sided first derivative. In either case ($\alpha = 2$ or $1 < \alpha < 2$), the boundary term enforces a no-flux condition at the point $y = 0$ in the state space.

The connection between probability and differential equations has profound consequences for mathematics [2, 8, 9, 36], and for its applications in science and engineering [21, 39, 40, 47], including a probabilistic method called particle tracking for solving fractional differential equations, by exploiting the associated Markov process [10, 54, 55]. More details on fractional calculus, and its connection to probability theory, may be found in the recent book of Meerschaert and Sikorskii [37]. Since fractional derivatives are nonlocal operators, the appropriate specification of boundary conditions requires a new approach [1, 16, 27, 29, 41, 50]. For example, one can apply the theory of Volterra integral equations [44] or general nonlocal operators [17]. The results in this paper can help clarify the meaning of reflecting boundary conditions for fractional diffusion.

We believe that this idea will find many useful applications, both inside and outside mathematics. As a first application, we show in Theorem 4.1 that a reflected stable process can be used to solve a fractional Cauchy problem, in which the usual first time derivative is replaced by a Caputo fractional derivative of order $0 < \beta < 1$.

Notation. We write $C_\infty[0,\infty)$ for the Banach space of continuous functions that vanish at infinity, i.e., $\lim_{x\to\infty} u(x) = 0$, with the uniform norm $\|u\| = \sup_{x\in[0,\infty)} |u(x)|$. Its topological dual is the space of (signed) Radon measures $\mathcal{M}_b[0,\infty)$, and by $\mathcal{M}_{ac}[0,\infty)$ we mean the absolutely continuous (with respect to Lebesgue measure) elements in $\mathcal{M}_b[0,\infty)$. On $\mathcal{M}_b[0,\infty)$ we use vague (weak-$*$) convergence; i.e., $\mu_n \to \mu$ if, and only if, $\int u\,d\mu_n \to \int u\,d\mu$ for all $u \in C_\infty[0,\infty)$. The subscripts c, b, ac, ∞ stand for 'compact support', 'bounded', 'absolutely continuous' and 'vanishing at infinity'. Fractional integrals and derivatives in the Riemann-Liouville sense are denoted by I^α and D^α, see (2.1)–(2.4) while Caputo derivatives are written as ∂^α, see (2.6). Finally, $\mathcal{L}g(s) = \mathcal{L}[g(t)](s) = \int_0^\infty e^{-st}\,g(t)\,dt$ denotes the usual Laplace transform, and $\mathcal{L}_{-\infty}g(s) = \mathcal{L}_{-\infty}[g(t)](s) = \int_{-\infty}^\infty e^{-st}\,g(t)\,dt$ is the bilateral Laplace transform.

2. The reflected stable process

Given a real number $\alpha > 0$ that is not an integer, define the positive Riemann-Liouville fractional integral

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x-y)y^{\alpha-1}\,dy = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(y)(x-y)^{\alpha-1}\,dy, \tag{2.1}$$

the negative Riemann-Liouville fractional integral

$$I_{-x}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty f(x+y)y^{\alpha-1}\,dy = \frac{1}{\Gamma(\alpha)}\int_x^\infty f(y)(y-x)^{\alpha-1}\,dy, \tag{2.2}$$

the positive Riemann-Liouville fractional derivative

$$D_x^\alpha f(x) := \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x f(y)(x-y)^{n-\alpha-1}\,dy, \tag{2.3}$$

and the negative Riemann-Liouville fractional derivative

$$D_{-x}^\alpha f(x) := \frac{d^n}{d(-x)^n} I_{-x}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^\infty f(y)(y-x)^{n-\alpha-1}\,dy, \tag{2.4}$$

where $n-1 < \alpha < n$. If $\alpha \in (1,2)$, then $n = 2$. See [7, 37, 45] for more details.

Let Z_t be the stochastic process defined in (1.2), where Y_t is a stable Lévy process with index $1 < \alpha < 2$ and characteristic function (1.1). Next we will show that Z_t is a conservative time-homogeneous Markov process whose (backward) semigroup $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ is strongly continuous ($\|T_t f - f\| \to 0$ as $t \downarrow 0$), contractive ($\|T_t f\| \leqslant \|f\|$), and analytic (the mapping $t \mapsto T(t)f$ has an analytic extension to the sectorial region $\{re^{i\theta} \in \mathbb{C} : r > 0, |\theta| < \alpha\}$ for some $\alpha > 0$) on the Banach space $X = C_\infty[0,\infty)$, and give a core for the generator. Recall that a core C_A of a closed linear operator A is a subset of its domain $\mathrm{D}(A)$ that is dense within the domain in the graph norm; i.e., for each $f \in \mathrm{D}(A)$ there exists a sequence $\{f_n\} \subset C_A$ such that $f_n \to f$ and $Af_n \to Af$.

Write

$$\begin{aligned} S_b := \Big\{ f \in C_\infty[0,\infty) \,:\, f'' \in C(0,\infty),\ f''(x) = O(1) \text{ as } x \to \infty, \\ f''(x) = O(x^{\alpha-2}) \text{ as } x \to 0,\ f' \in C_b(0,\infty),\ f'(0+) = 0 \Big\} \end{aligned} \tag{2.5}$$

and denote by ∂_x^α the (positive) Caputo fractional derivative of order $\alpha > 0$, which can be defined by

$$\partial_x^\alpha f(x) = I_x^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-y)^{n-1-\alpha} f^{(n)}(y)\,dy \tag{2.6}$$

where $n-1 < \alpha < n$ and $f^{(n)}$ is the nth derivative of f. The Caputo fractional derivative differs from the Riemann-Liouville form (2.3) because the operations of differentiation and (fractional) integration do not commute in general. For example, when $0 < \alpha < 1$ we have

$$\partial_x^\alpha f(x) = D_x^\alpha f(x) - f(0)\frac{x^{-\alpha}}{\Gamma(1-\alpha)} \tag{2.7}$$

for suitably nice bounded functions (e.g., see [37, p. 39]).

Theorem 2.1. *Let Z_t denote the reflected process* (1.2) *where Y_t is a stable Lévy process with index $\alpha = 1/\beta \in (1,2)$ and characteristic function* (1.1)*. Then Z_t is a Feller process, the transition semigroup $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ on $C_\infty[0,\infty)$ is analytic, with generator $Af = \partial_x^\alpha f$ for all $f \in C_A = \{f \in S_b : \partial_x^\alpha f \in C_\infty[0,\infty)\}$, where ∂_x^α is the Caputo fractional derivative* (2.6)*, and C_A is a core of A.*

Proof. Define the running infimum $I_t = \inf\{Y_s : 0 \leqslant s \leqslant t\}$ and the running supremum $S_t = \sup\{Y_s : 0 \leqslant s \leqslant t\}$. Let $\hat{Y}_t = -Y_t$ denote the dual process, and let $\hat{I}_t$ and $\hat{S}_t$ denote the running infimum and supremum of $\hat{Y}_t$, respectively. Since $\hat{Y}_t$ is also a Lévy process, it follows from [12, Section VI.1, Proposition 1] that $\hat{S}_t - \hat{Y}_t$ is a Feller process, and since $\hat{S}_t = -I_t$, it follows that $\hat{S}_t - \hat{Y}_t = -I_t + Y_t = Z_t$. Hence Z_t is a Feller process.

It follows from [11, Proposition 4] that $C_A \subset \mathrm{D}(A)$ and $Af = \partial_x^\alpha f$ for all $f \in C_A$. Note that the extension from f'' bounded to $|f''(x)| = O(x^{\alpha-2})$ at $x = 0+$ is also mentioned in the proof.

Since A generates a strongly continuous contraction semigroup, the resolvent operators $R(\lambda, A) := (\lambda - A)^{-1}$ exist for all $\operatorname{Re}\lambda > 0$ and they are bounded operators. We will now show that

$$R(\lambda, A)g(x) = -\alpha x^{\alpha-1}E_\alpha'(\lambda x^\alpha) \star g(x) + \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}}E_\alpha(\lambda x^\alpha) \tag{2.8}$$

for all $g \in C_\infty[0,\infty)$, where $\mathcal{L}g$ is the Laplace transform, $\star$ is the convolution operator, and the Mittag-Leffler function $E_\alpha(x) = \sum_{n=0}^\infty x^n/\Gamma(1+\alpha n)$.

Let $R_{\lambda,g}$ denote the right-hand side of (2.8). Since

$$\mathcal{L}[E_\alpha(\lambda x^\alpha)](s) = \frac{s^{\alpha-1}}{s^\alpha - \lambda} \quad \text{and} \quad \mathcal{L}[\alpha x^{\alpha-1}E_\alpha'(\lambda x^\alpha)](s) = \frac{1}{s^\alpha - \lambda},$$

see e.g. [22, 32], it follows that

$$\mathcal{L}R_{\lambda,g}(s) = \frac{\mathcal{L}g(s)}{\lambda - s^\alpha} - \frac{s^{\alpha-1}}{\lambda - s^\alpha}\frac{\lambda^{1/\alpha}\mathcal{L}g(\lambda^{1/\alpha})}{\lambda} \tag{2.9}$$

Using the fact that $[f \star g]'(x) = [f' \star g](x) + f(0)g(x)$ it follows that $R_{\lambda,g}$ is twice differentiable for any $g \in C_\infty[0,\infty) \cap C^2[0,\infty)$. Equation (2.6) implies that for any $f \in C^2[0,\infty)$ we have

$$\mathcal{L}[\partial_x^\alpha f](s) = s^{\alpha-2}\left(s^2\mathcal{L}f(s) - sf(0) - f'(0)\right) = s^\alpha \mathcal{L}f(s) - s^{\alpha-1}f(0).$$

Taking the Laplace transform of $\lambda R_{\lambda,g} - \partial_x^\alpha R_{\lambda,g}$, we therefore obtain that

$$\begin{aligned} \mathcal{L}\left[\lambda R_{\lambda,g} - \partial_x^\alpha R_{\lambda,g}\right](s) &= \lambda \mathcal{L} R_{\lambda,g}(s) - s^\alpha \mathcal{L} R_{\lambda,g}(s) + s^{\alpha-1} R_{\lambda,g}(0) \\ &= \mathcal{L} g(s) - \frac{s^{\alpha-1}}{\lambda^{1-1/\alpha}} \mathcal{L} g(\lambda^{1/\alpha}) + s^{\alpha-1} R_{\lambda,g}(0) \\ &= \mathcal{L} g(s) \end{aligned} \tag{2.10}$$

for any $g \in C_\infty[0,\infty) \cap C^2[0,\infty)$, and hence $R(\lambda, A)g = R_{\lambda,g}$. By continuous extension, (2.8) holds for all $g \in C_\infty[0,\infty)$.

To show that C_A is a core, note that for $g \in C_\infty^2[0,\infty) := C^2[0,\infty) \cap C_\infty[0,\infty)$,

$$\partial_x^\alpha R(\lambda, A)g = \lambda R(\lambda, A)g - g \in C_\infty[0,\infty)$$

and hence $R(\lambda, A)C_\infty^2[0,\infty) \subset C_A$. Pick $f \in \mathrm{D}(A)$ and $g = \lambda f - Af$. Since $C_\infty^2[0,\infty)$ is dense in $C_\infty[0,\infty)$, there exists a sequence $\{g_n\} \subset C_\infty^2[0,\infty)$ with $g_n \to g$. Thus, $f_n = R(\lambda, A)g_n \to f$ and $Af_n = \lambda f_n - g_n \to \lambda f - g = Af$. Since $f_n \in R(\lambda, A)C_\infty^2[0,\infty) \subset C_A$, we see that C_A is a core.

Finally we show that $\{T_t\}_{t\geqslant 0}$ is an analytic semigroup. Since $R(\lambda, A)$ is a bounded operator for all $\operatorname{Re}\lambda > 0$, a general result from the theory of semigroups [3, Corollary 3.7.12] shows that $\{T_t\}$ is analytic if for some some $M > 0$ we have

$$\|\lambda R(\lambda, A)g\| \leqslant M\|g\| \tag{2.11}$$

for all $\operatorname{Re}\lambda > 0$ and all $g \in C_\infty[0,\infty)$. Then the result follows from Lemma 5.2, and this completes the proof. □

Remark 2.2. Patie and Simon [42] show that the reflected stable process Z_t in Theorem 2.1 has the backward generator

$$Af(x) = f'(0)\frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^x f''(x-y)\frac{y^{1-\alpha}}{\Gamma(2-\alpha)}\,dy. \tag{2.12}$$

They also give the exact domain of the generator [42, Proposition 2.2]. If $f \in S_b$, then $f'(0) = 0$, and Af reduces to the Caputo fractional derivative (2.6).

In view of Theorem 2.1, $T_tf(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ is a strongly continuous, analytic semigroup on the Banach space $X := C_\infty[0,\infty)$ with the supremum norm, with generator $Af(x) = \partial_x^\alpha f(x)$ for $f \in S_b$ such that $\partial_x^\alpha f \in X$. The dual (or adjoint) semigroup T_t^* is defined on the dual space $X^* = \mathcal{M}_b[0,\infty)$ of finite signed Radon measures on $[0,\infty)$ equipped with the total variation norm: Given a measure $\mu \in \mathcal{M}_b[0,\infty)$, use the Jordan decomposition to write $\mu = \mu^+ - \mu^-$ uniquely as a difference of two positive measures, and define $\|\mu\| = \mu^+[0,\infty) + \mu^-[0,\infty)$. The dual semigroup satisfies

$$\int T_tf(x)\mu(dx) = \int f(x)[T_t^*\mu](dx) \tag{2.13}$$

for all $f \in C_\infty[0,\infty)$ and all $\mu \in \mathcal{M}_b[0,\infty)$. See [19, Section 2.5] for more details. In probabilistic terms, since $T_tf(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ for this time-homogeneous

Markov process, equation (2.13) implies that

$$\int T_t f(x)\mu(dx) = \int \mathbb{E}[f(Z_{t+s})|Z_s = x]\mu(dx)$$
$$= \int f(y)P_t(dy,\mu) = \int f(y)[T_t^*\mu](dy)$$

where $P_t(y,\mu) = \int P(y,x,t)\mu(dx)$ and $P(y,x,t) = \mathbb{P}[Z_{t+s} \leqslant y|Z_s = x]$ is the transition probability distribution of the Markov process Z_t. Hence, if μ is the probability distribution of Z_s, then $T_t^*\mu(dy) = P_t(dy,\mu)$ is the probability distribution of Z_{t+s}. The dual semigroup is also called the *forward semigroup* associated with the Markov process Z, since it maps the probability distribution forward in time.

Next we will compute the generator A^* of the forward semigroup. This is the adjoint of the generator A of the backward semigroup, in the sense that

$$\int Af(x)\mu(dx) = \int f(x)[A^*\mu](dx)$$

for all $f \in \mathrm{D}(A)$ and $\mu \in \mathrm{D}(A^*)$. Theorem 2.3 will show that every measure $\mu \in \mathrm{D}(A^*)$ has a Lebesgue density $g \in L^1[0,\infty)$, so that $\mu(dy) = g(y)\,dy$, and that the adjoint $A^*g := A^*\mu$ of the positive fractional Caputo derivative $Af(x) = \partial_x^\alpha f(x)$ in our setting is the negative Riemann-Liouville fractional derivative $A^*g(y) = D_{-y}^\alpha g(y)$ using (2.4).

The forward semigroup T_t^* of a Markov process is not, in general, strongly continuous on $\mathcal{M}_b[0,\infty)$. That is, there exist measures μ such that $T_t^*\mu \not\to \mu$ in the total variation norm as $t \downarrow 0$. For example, if T_t^* is the forward semigroup associated with the diffusion equation $\partial_t p = \partial_x^2 p$, and $\mu = \delta_0$ is a point mass at the origin, then $T_t^*\mu$ is a Gaussian probability measure with mean 0 and variance $2t$ for all $t > 0$, and since $\mu\{0\} = 1$ and $T_t\mu\{0\} = 0$ for all $t > 0$, we have $\|T_t\mu - \mu\| = 1$ for all $t > 0$ in the total variation norm.

To handle this situation, we define the *sun dual space* of $X := C_\infty[0,\infty)$ as

$$X^\odot := \{\mu \in X^* : \lim_{t\downarrow 0} \|T_t^*\mu - \mu\| = 0\},$$

a closed subspace of $X^* = \mathcal{M}_b[0,\infty)$ on which the forward semigroup is strongly continuous. It follows from basic semigroup theory [19, Section 2.6] that for $\mu \in X^\odot$, $T_t^*\mu \in X^\odot$ for all $t \geqslant 0$, and $X^\odot = \overline{\mathrm{D}(A^*)}$. The restriction of $\{T_t^*\}_{t\geqslant 0}$ to $X^\odot$ is called the *sun dual semigroup* $\{T_t^\odot\}_{t\geqslant 0}$ with generator $A^\odot\mu = A^*\mu$ for all $\mu \in \mathrm{D}(A^\odot)$, where

$$\mathrm{D}(A^\odot) = \{\mu \in \mathrm{D}(A^*) : A^*\mu \in X^\odot\}. \tag{2.14}$$

For the reflected stable process, we will show in Theorem 2.3 that $C_\infty^\odot[0,\infty)$ is the space of absolutely continuous elements of $\mathcal{M}_b[0,\infty)$,

$$\mathcal{M}_{ac}[0,\infty) = \big\{\mu \in \mathcal{M}_b[0,\infty) \,:\, \mu(dy) = g(y)\,dy \text{ for some } g \in L^1[0,\infty)\big\}.$$

and we will derive the forward equation of the reflected stable process on the sun-dual space. For a general bounded measure $\mu \in \mathcal{M}_b[0,\infty)$, we will then prove in Corollary 2.5 that $T_t^*\mu$ can be computed as the vague limit of $T_t^\odot\mu_n$, where $\mu_n \to \mu$ vaguely, and $\mu_n \in C_\infty^\odot[0,\infty)$ for all n.

Theorem 2.3. *Let Z_t denote the Feller process* (1.2)*, where Y_t is a stable Lévy process with index $\alpha = 1/\beta \in (1,2)$ and characteristic function* (1.1)*, with (backward) semigroup $T_t f(x) = \mathbb{E}[f(Z_{t+s})|Z_s = x]$ on $C_\infty[0,\infty)$. Then $C_\infty^\odot[0,\infty) = \mathcal{M}_{ac}[0,\infty)$ and the generator $A^\odot g := A^\odot \mu$ of the sun-dual semigroup $\{T^\odot(t)\}_{t\geqslant 0}$ is given by*

$$A^\odot g(y) = D_{-y}^\alpha g(y) \tag{2.15}$$

with domain $\mathrm{D}(A^\odot) = \{g \in L^1[0,\infty) : D_{-y}^\alpha g(y) \in L^1[0,\infty), D_{-y}^{\alpha-1} g(0) = 0\}$.

Proof. Suppose $A^*\mu = \nu \in \mathcal{M}_b[0,\infty)$ for some $\mu \in \mathcal{M}_b[0,\infty)$, so that $\int Af(x)\mu(dx) = \int f(x)\nu(dx)$ for all $f \in \mathrm{D}(A)$. Set $v(x) := \nu[0,x]$ for $x \geqslant 0$ and $v(x) = 0$ for $x < 0$. If $f \in S_b$ with $\partial_x^\alpha f \in C_\infty[0,\infty)$, then it follows from Theorem 2.1 that $f \in \mathrm{D}(A)$ and $Af = \partial_x^\alpha f$. It is obvious from the Definition (2.6) that $\partial_x^{\alpha-1} f'(x) = \partial_x^\alpha f(x)$. Let

$$I_x^\alpha f(x) = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} f(y)\,dy$$

denote the positive Riemann-Liouville fractional integral (2.1) of order $\alpha > 0$ for a function $f \in C_\infty[0,\infty)$, and apply the general formula [7, Eq. (1.21)] $I_x^{\alpha-1}\partial_x^{\alpha-1} f'(x) = f'(x) - f'(0)$ to see that $f'(x) = I_x^{\alpha-1}\partial_x^{\alpha-1} f'(x) = I_x^{\alpha-1}\partial_x^\alpha f(x) = I_x^{\alpha-1} Af(x)$. Since $f(x) \to 0$ as $x \to \infty$, and $v(0) = 0$ for $x < 0$, we can apply the integration by parts formula [23, Theorem 19.3.13]

$$\int_a^b f(x)\nu(dx) = f(b)v(b) - f(a)v(a) - \int_a^b v(x) f'(x) dx$$

with $a < 0$, and then let $b \to \infty$, to see that

$$\int_0^\infty f(x)\nu(dx) = -\int_0^\infty f'(x)v(x)\,dx.$$

Thus, for all $f \in S_b$ with $\partial_x^\alpha f \in C_\infty[0,\infty)$, a Fubini argument yields

$$\begin{aligned}\int_0^\infty f(x)\nu(dx) &= -\int_0^\infty \int_0^x \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} Af(y)\,dy\,v(x)\,dx \\ &= -\int_0^\infty \int_y^\infty \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} v(x)\,dx\,Af(y)\,dy \\ &= \int_0^\infty Af(y)\,\mu(dy).\end{aligned} \tag{2.16}$$

Next we will show that $S := \{Af : f \in S_b \text{ with } \partial_x^\alpha f \in C_\infty[0,\infty)\}$ is dense in $C_\infty[0,\infty)$, and then it will follow that any measure $\mu \in \mathrm{D}(A^*)$ has a Lebesgue density

$$g(y) = -\int_y^\infty \frac{(x-y)^{\alpha-2}}{\Gamma(\alpha-1)} v(x)\,dx = -I_{-y}^{\alpha-1} v(y) \tag{2.17}$$

where $A^*\mu = \nu$ and $v(x) = \nu[0,x]$. Let $C_c^\infty[0,\infty)$ denote the space of smooth functions with compact support, i.e., such that $h(x) = 0$ for all $x > M$, for some $M > 0$. It is not hard to check that the space $Q = \{h \in C_c^\infty[0,\infty) : \int h = 0\}$ is dense in $C_\infty[0,\infty)$. Then we certainly have

$$\lim_{x\to\infty} I_x^\alpha h(x) = \lim_{x\to\infty} \int_0^M \frac{(x-s)^{\alpha-1} - x^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds = 0,$$

and therefore, the function $f(x) = I_x^\alpha h(x)$ is an element of $C_\infty[0,\infty)$ for any $h \in Q$. Elementary estimates suffice to check that $f \in S_b$ as well. Since the positive Caputo derivative is a left inverse of the positive Riemann-Liouville integral [7, Eq. (1.21)], we also have $\partial_x^\alpha f(x) = \partial_x^\alpha I_x^\alpha h(x) = h(x) \in C_\infty[0,\infty)$. Hence $h = Af \in S$ for all $h \in Q$, and thus $Q \subseteq S$. Now for any $f \in C_\infty[0,\infty)$ there exists a sequence $f_n \to f$ in the supremum norm, with $f_n \in S$ for all n. Then some simple estimates can be used to verify that $\int f(y)\mu(dy) = \int f(y)g(y)\,dy$, and it follows that the measure μ has the Lebesgue density g. Hence we can identify $\mathrm{D}(A^*)$ with a subspace of $L^1[0,\infty)$.

Next we will show that this subspace is dense in $L^1[0,\infty)$. Define

$$\phi_n(x) := \tfrac{1}{\Gamma(\alpha)} \left(\tfrac{1}{n} - x\right)^{\alpha-1} \mathbb{1}_{[0,1/n)}(x),$$

and note that $D_{-x}^{\alpha-1}\phi_n(x) \equiv 1$ for $x \in (0, 1/n)$ by a straightforward computation. Note that the set $U := \{g(x) - [D_{-x}^{\alpha-1} g(0)]\phi_n : g \in C_c^\infty[0,\infty), n \in \mathbb{N}\}$ is dense in $L^1[0,\infty)$, and that $D_{-x}^\alpha g(x) \in L^1[0,\infty)$ for all $g \in U$. Furthermore, U is a subset of

$$G = \left\{g \in L^1[0,\infty) \,:\, D_{-x}^\alpha g \in L^1[0,\infty),\ D_{-x}^{\alpha-1} g(0) = 0\right\}.$$

For any $g \in G$ and $f \in S_b$ with $\partial_x^\alpha f \in C_\infty[0,\infty)$, we have $D_{-x}^{\alpha-1} g(0) = 0$ and $f'(0) = 0$. Then Theorem 2.1, a Fubini argument, and integration by parts (twice) using equation (2.4) yields

$$\begin{aligned}
\int_0^\infty Af(y)\, g(y)\, dy &= \int_0^\infty \int_0^y \frac{(y-x)^{1-\alpha}}{\Gamma(2-\alpha)} f''(x)\, dx\, g(y)\, dy \\
&= \int_0^\infty \int_x^\infty \frac{(y-x)^{1-\alpha}}{\Gamma(2-\alpha)} g(y)\, dy\, f''(x)\, dx \\
&= \int_0^\infty I_{-x}^{2-\alpha} g(x) f''(x)\, dx \\
&= \int_0^\infty D_{-x}^{\alpha-1} g(x) f'(x)\, dx \\
&= \int_0^\infty f(x)\, D_{-x}^\alpha g(x)\, dx.
\end{aligned} \tag{2.18}$$

Furthermore, as $\{f \in S_b : Af \in C_\infty[0,\infty)\}$ is a core, for all $f \in \mathrm{D}(A)$ there exists a sequence f_n in the core such that $f_n \to f$ and $Af_n \to Af$. Hence (2.18) holds for all $f \in \mathrm{D}(A)$ and therefore $g \in \mathrm{D}(A^*)$ and $A^* g = D_{-x}^\alpha g$ for any $g \in G$. Since U is dense in $L^1[0,\infty)$, and $U \subseteq G \subseteq \mathrm{D}(A^*)$, it follows that $\mathrm{D}(A^*)$ is dense in $L^1[0,\infty)$. Since $C_\infty^\odot[0,\infty)$ is the smallest closed set containing $\mathrm{D}(A^*)$ by definition, and since $L^1[0,\infty)$ is a closed subspace of $\mathcal{M}_b[0,\infty)$, we have shown that $L^1[0,\infty) = C_\infty^\odot[0,\infty)$.

Equation (2.14) implies that $\nu = A^*\mu$ is an element of $C_\infty^\odot[0,\infty)$ for any $\mu \in \mathrm{D}(A^\odot)$, and therefore, we have $\nu(dx) = h(x)\,dx$ for some $h \in L^1[0,\infty)$, as well as $\mu(dy) = g(y)\,dy$ for some $g \in L^1[0,\infty)$. Since $v(x) = \nu[0,x]$, it follows that $h(x) = v'(x)$. Since

$D^{\alpha-1}_{-y}$ is a left inverse of $I^{\alpha-1}_{-y}$ in general, it follows from (2.17) that $v(y) = -D^{\alpha-1}_{-y}g(y)$. Then we have

$$h(x) = v'(x) = \frac{d}{dx}\left[-D^{\alpha-1}_{-x}g(x)\right] = D^{\alpha}_{-x}g(x) = A^*g(x) \tag{2.19}$$

for all $g \in \mathrm{D}(A^{\odot})$, which proves the generator formula (2.15). Equation (2.19) also shows that $D^{\alpha}_{-x}g(x) \in L^1[0,\infty)$ for all $g \in \mathrm{D}(A^{\odot})$, and since v is continuous with $v(0) = 0$, it follows that $D^{\alpha-1}_{-x}g(0) = v(0) = 0$ for all $g \in \mathrm{D}(A^{\odot})$. This proves that $\mathrm{D}(A^{\odot}) \subseteq G$. Since $\mathrm{D}(A^{\odot})$ is defined as the set of $g \in \mathrm{D}(A^*)$ such that $A^*g \in L^1[0,\infty)$, it follows from (2.18) that $g \in \mathrm{D}(A^{\odot})$ for all $g \in G$, so that $G \subseteq \mathrm{D}(A^{\odot})$ as well, which completes the proof. □

Theorem 2.3 establishes the forward equation of the reflected stable process Z_t, for certain initial conditions. It shows that, for any initial condition $\mu_0(dx) = g(x)dx$ where $g \in L^1[0,\infty)$, $\mu_t := T^{\odot}\mu$ solves the Cauchy problem

$$\partial_t\mu(t) = A^{\odot}\mu(t); \quad \mu(0) = \mu_0$$

on the sun-dual space. This implies that, for any probability density $p_0(x)$ such that $p_0(x) = 0$ for $x < 0$, the function $p(x,t) = T^{\odot}p_0(x)$ solves the forward equation

$$\partial_t p(x,t) = D^{\alpha}_{-x}p(x,t); \quad p(x,0) = p_0(x), \quad D^{\alpha-1}_{-x}p(x,t)\Big|_{x=0} \equiv 0. \tag{2.20}$$

The next two results will allow us to compute the transition probability density $y \mapsto p(x,y,t)$ of the time-homogeneous Markov process $y = Z_{t+s}$ for any initial state $x = Z_s$, by applying Theorem 2.3 to a sequence of initial conditions $\mu_n \in X^{\odot}$ such that $\mu_n \to \delta_x$. The first result shows that the transition density exists.

Corollary 2.4. *The semigroup $T^{\odot}_t$ is a strongly continuous bounded analytic semigroup on $L^1(\mathbb{R}_+)$ and the transition probability distributions $T^*_t\delta_x$ have smooth densities $y \mapsto p(x,y,t)$ for all $t > 0$ and all $x \geqslant 0$.*

Proof. It is well-known that the spectra of A and A^* coincide, and $R(\lambda, A^*) = R(\lambda, A)^*$ for all λ in the resolvent set of A (and A^*). Therefore, since A is a sectorial operator, being the generator of a bounded analytic semigroup, it follows that A^* is a sectorial operator as well, and hence A^* generates a bounded analytic semigroup (not necessarily strongly continuous at 0) on $C^*_\infty[0,\infty) = \mathcal{M}_b[0,\infty)$ by [3, Theorem 3.7.1] which coincides with T^*_t for all $t > 0$ by the uniqueness of the Laplace transform. Therefore, the restriction of A^* to $\overline{\mathrm{D}(A^*)} = L^1[0,\infty)$ (i.e., the operator $A^{\odot}$) generates a strongly continuous bounded analytic semigroup $T^{\odot}_t$ on $L^1[0,\infty)$ by [3, Remark 3.7.13]. Furthermore, [3, Remark 3.7.20] shows that for all $n \in \mathbb{N}$ and $t > 0$ we have that $T^*_t\mu \in \mathrm{D}((A^*)^n)$ for all $\mu \in \mathcal{M}_b[0,\infty)$ and that $T^{\odot}_t f \in \mathrm{D}((A^{\odot})^n)$ for all $f \in L^1[0,\infty)$. Since $\mathrm{D}(A^*) \subset L^1[0,\infty)$, it follows that $T^*_s\mu \in L^1[0,\infty)$ for all $s > 0$ and $\mu \in \mathcal{M}_b[0,\infty)$. Therefore for all $t > 0$ and $\mu \in \mathcal{M}_b[0,\infty)$ we have

$$T^*_t\mu = T^*_{\frac{t}{2}}T^*_{\frac{t}{2}}\mu = T^{\odot}_{\frac{t}{2}}T^*_{\frac{t}{2}}\mu \in \mathrm{D}((A^{\odot})^n)$$

for all $n \in \mathbb{N}$. Thus by taking $\mu = \delta_x$ it follows that $T^*_t\delta_x \in \mathrm{D}((A^{\odot})^n)$ for all n. Using Theorem 2.3, we have that $\mathrm{D}(A^{\odot}) = \{g \in L^1[0,\infty) : D^{\alpha}_{-y}g(y) \in L^1[0,\infty), D^{\alpha-1}_{-y}g(0) = 0\}$, and then it follows that $\left(D^{\alpha}_{-y}\right)^n T^*_t\delta_x \in L^1[0,\infty)$ for all n. In particular, the transition probability distribution $T_t^*\delta_x$ has a density function $y \mapsto p(x,y,t)$ for all $t > 0$ and all $x \geqslant 0$. To see that this function is smooth note that any positive integer m can be written in the form $m = n\alpha - \beta$ for some integer $n \geqslant 1$ and some positive real number $\beta < \alpha$. A straightforward calculation shows that for $f \in \mathrm{D}((A^{\odot})^n)$ we have $\left(D_{-y}^{\alpha}\right)^n f = D_{-y}^{n\alpha} f$ and $I_{-y}^{\beta} D_{-y}^{n\alpha} f = D_{-y}^{n\alpha-\beta} f \in L^1[0,\infty)$ for all $\beta < \alpha$ and $n \geqslant 1$. Since $D_{-y}^m f = (-1)^m (d/dy)^m f$, we have $(d/dy)^m p(x,y,t) \in L^1[0,\infty)$ for all m. □

Corollary 2.5. *Let* $\{\mu_n\} \subset C_\infty^{\odot}[0,\infty) = \mathcal{M}_{ac}[0,\infty)$ *such that* $\mu_n \to \mu$ *vaguely as* $n \to \infty$ *for some* $\mu \in \mathcal{M}_b[0,\infty)$*, then* $T_t^{\odot}\mu_n \to T_t^*\mu$ *vaguely as* $n \to \infty$.

Proof. Since $T_t^{\odot}\mu_n = T_t^*\mu_n$ for all $n \geqslant 1$, for all $\phi \in C_\infty[0,\infty)$ we have

$$\begin{aligned} \int \phi(x)[T_t^{\odot}\mu_n](dx) &= \int \phi(x)[T_t^*\mu_n](dx) = \int [T_t\phi](x)\mu_n(x)dx \\ &\xrightarrow{n\to\infty} \int [T_t\phi](x)\mu(dx) = \int \phi(x)[T_t^*\mu](dx), \end{aligned} \tag{2.21}$$

and the result follows. □

In Section 3, we will apply Corollary 2.5 to compute the transition densities of the reflected stable process to any desired degree of accuracy, by solving the forward equation numerically. For any initial state $Z_s = x$, we will approximate the initial condition $\mu_0 = \delta_x$ in the numerical method by a sequence of measures μ_n with L^1-densities, and then Corollary 2.5 guarantees that the resulting solutions converge to the transition density in the supremum norm as $n \to \infty$.

Remark 2.6. Using integration by parts, one can write the backward generator in the form of an integro-differential operator (e.g., see Jacob [26])

$$Af(x) = b(x)f'(x) + \int \left[f(x+y) - f(x) - yf'(x)\right]\phi(x,dy), \tag{2.22}$$

with coefficients

$$\begin{aligned} b(x) &= \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} \quad \text{and} \\ \phi(x,dy) &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}|y|^{-1-\alpha}dy\mathbb{1}_{(-x,0)}(y) + \frac{\alpha-1}{\Gamma(2-\alpha)}x^{-\alpha}\varepsilon_{-x}(dy). \end{aligned} \tag{2.23}$$

The jump intensity $\phi(x,dy)$ describes the behavior of the process Z_t, which truncates jumps of the stable process Y_t starting at the point $x > 0$ in the state space, so that a jump (they are all negative) of size $|y| > x$ is changed to a jump of size x. This keeps the sample paths of Z_t inside the half-line $[0,\infty)$. Since the drift $b(x)$ is unbounded, the existence of a Markov process Z_t with generator (2.22) would not follow from general theory (e.g., see [20, Section 4.5] or [48, Section 3]). Hence the reflected stable process Z_t is an interesting example of a Markov process with unbounded drift coefficient.

3. Transition density of the reflected stable process

To the best of our knowledge, there is no known analytical formula for the transition density $y \mapsto p(x;t,y)$ of the reflected stable process $y = Z_{t+s}$ started at $x = Z_s > 0$. In this section, we compute and plot this transition density, by numerically solving the associated forward equation (2.20). The existence and smoothness of these transition densities is guaranteed by Corollary 2.4. Corollary 2.5 shows that $T_t^{\odot} g_n(y)dy \to T_t^* \delta_x(dy)$ in the supremum norm as $n \to \infty$ for any sequence of functions $g_n \in L^1[0,\infty)$ such that $g_n(y)dy \to \delta_x(y)dy$ (vague convergence). We take $g_n(y) = n\ \mathbb{1}_{[x,x+1/n]}(y)$. Then the solutions $T_t^{\odot} g_n$ provide estimates of the transition density to any desired degree of accuracy. Theorem 2.3 shows that $T_t^{\odot} g_n$ solves the forward equation (2.20). Hence, we can compute the transition densities of the stable process by solving the forward equation numerically, with this initial condition.

In order to compute the probability density $p(x,y,t)$ numerically, we consider the fractional boundary value problem

$$\partial_t u(y,t) = D^{\alpha}_{-y} u(y,t); \quad u(y,0) = \delta_x(y); \quad D^{\alpha-1}_{-y} u(0,t) = 0. \tag{3.1}$$

We develop forward-stepping numerical solutions $u_h(y_i,t)$ that estimate $u(y_i,t)$ at locations $y_i = ih$ for $i = 0,1,\dots,N$ over an interval $[0, y_{max}]$ in the state space, where y_{max} is chosen large enough and h is chosen small enough so that enlarging the domain further, or making the step size smaller, has no appreciable effect on the computed solutions (e.g., the resulting graph does not visibly change). We approximate the delta function initial condition $u(y,0) = \delta_x(y)$ by setting $u_h(y_i,0) = 1/h$ for $y_i = x$ and $u_h(y_i,0) = 0$ otherwise, a numerical representation of the initial condition $g_n(y) = n\ \mathbb{1}_{[x,x+1/n]}(y)$ with $h = 1/n$.

Numerical methods for fractional differential equations are an active area of research. One important finding [34, 35] is that a shifted version of the Grünwald finite difference formula (1.4) for the fractional derivative is required to obtain a stable, convergent method. Hence we approximate

$$D^{\alpha}_{-y} u(y_i,t) \approx \frac{1}{h^{\alpha}} \sum_{k=0}^{N+1-i} w_k^{\alpha} u_h(y_{i+k-1},t) \quad \text{where} \quad w_k^{\alpha} := (-1)^k \binom{\alpha}{k} \tag{3.2}$$

for $i \geqslant 2$. Note that this approximation of $D^{\alpha}_{-y} u(y_i,t)$ does not depend on the value of u_h at the boundary $y_0 = 0$. We enforce the boundary condition at $y_0 = 0$ at each step; i.e., we set

$$u_h(y_0,t) = -\sum_{k=1}^{N} w_k^{\alpha-1} u_h(y_k,t)$$

so that

$$D^{\alpha-1}_{-y} u(y_0,t) \approx \sum_{k=0}^{N} w_k^{\alpha-1} u_h(y_k,t) = 0$$

since $w_0^{\alpha-1} = 1$. Finally, for $i = 1$ we approximate

$$
\begin{aligned}
D_{-y}^{\alpha} u(y_1, t) \approx & \frac{1}{h^\alpha} \sum_{k=0}^{N} w_k^\alpha u_h(y_{1+k-1}, t) \\
= & \frac{1}{h^\alpha} \left(\sum_{k=1}^{N} w_k^\alpha u_h(y_k, t) - \sum_{k=1}^{N} w_k^{\alpha-1} u_h(y_k, t) \right) \\
= & -\frac{1}{h^\alpha} \sum_{k=1}^{N} w_{k-1}^{\alpha-1} u_h(y_k, t)
\end{aligned}
\tag{3.3}
$$

using an elementary identity for fractional binomial coefficients $w_k^\alpha - w_k^{\alpha-1} = -w_{k-1}^{\alpha-1}$.

This leads to the following linear system of ordinary differential equations,

$$
\frac{d}{dt} \begin{pmatrix} u_h(y_1, t) \\ \vdots \\ \vdots \\ \vdots \\ u_h(y_N, t) \end{pmatrix} = \frac{1}{h^\alpha} \begin{pmatrix} -1 & \alpha - 1 & \dots & \dots & -w_{N-1}^{\alpha-1} \\ 1 & -\alpha & w_2^\alpha & \dots & w_{N-1}^\alpha \\ 0 & 1 & \ddots & \ddots & w_{N-2}^\alpha \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -\alpha \end{pmatrix} \begin{pmatrix} u_h(y_1, t) \\ \vdots \\ \vdots \\ \vdots \\ u_h(y_N, t) \end{pmatrix}
\tag{3.4}
$$

whose solution can be approximated using any numerical ODE solver (indeed, this is a linear system of the form $u' = Au$, so it has a solution $u(t) = e^{tA}u(0)$ for any initial condition). The resulting numerical solutions with $y_{\max} = 12$, $h = 0.01$, starting points $x = 0, 1, 2, 4$, and $\alpha = 1.2, 1.8$ are depicted in Figures 1 and 2, where each frame represents a snapshot at times $t = 0.5, 1$ and 2 respectively. The L^1-error of the numerical solutions for $x = 0$ decays linearly with h. For $h = 0.01$ the L^1-error is less than 0.05 for $\alpha = 1.2$, and less than 0.004 for $\alpha = 1.8$, for every case plotted. A short MATLAB code to compute the numerical solution is included in the Appendix.

Remark 3.1. It is interesting to note that the matrix in (3.4) is essentially the rate matrix of a discrete state Markov process in continuous time. Extending the state space to $N = \infty$, we obtain a Markov process Z_t^h on the state space $\{ih : i > 0\}$ that approximates the reflected stable process, with $u_h(x_i, t) = \mathbb{P}(Z_t^h = ih)$. The transition rate from state ih to state jh for $i > j > 1$ is $w_{i-j+1}^\alpha \approx \alpha(\alpha-1)(i-j)^{-\alpha-1}/\Gamma(2-\alpha)$, the jump intensity of the stable process Y_t, in view of [37, Eq. (2.5)]. The transition rate from state ih for $i > 1$ to state jh for $j = 1$, in the first row of the rate matrix, is $w_{i-1}^{\alpha-1} \approx i^{-\alpha}/\Gamma(1-\alpha)$, the rate at which the process Y_t would jump into the negative half-line. This can be computed as $\phi(-\infty, -ih)$ where $\phi(dy) = \alpha(\alpha-1)|y|^{-1-\alpha}dy/\Gamma(2-\alpha)$ is the Lévy measure of the process Y_t, e.g., see [37, Proposition 3.12].

Remark 3.2. As noted in the introduction, the fractional boundary condition in (3.1) is a natural extension of the boundary condition (1.3) for Brownian motion on the half-line. The fractional boundary condition in (3.1) can be written in Grünwald finite difference form using [37, Proposition 2.1] to arrive at (1.4).

Remark 3.3. Bernyk, Dalang and Peskir [11, Appendix] computed the backward generator of a general reflected stable Lévy process. Caballero and Chaumont [14, Theorem

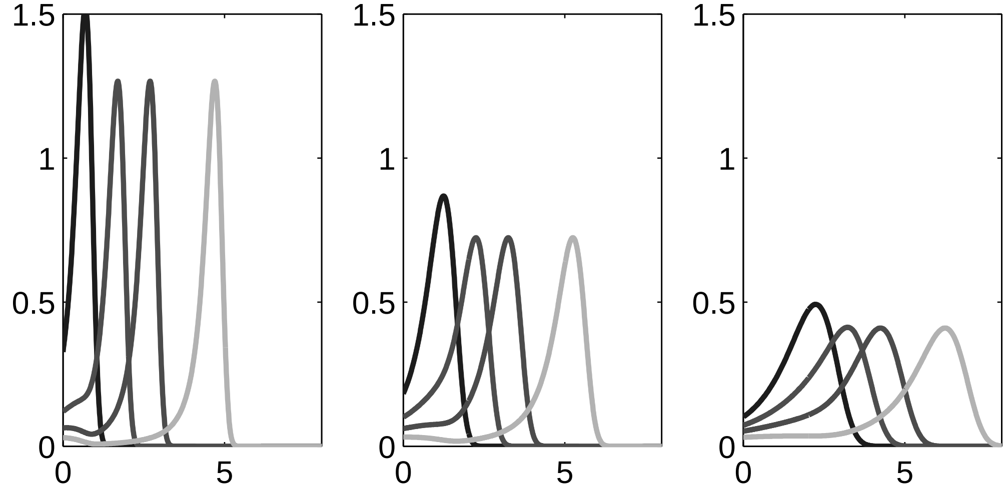

Figure 1. The transition densities of $p(x, y, t)$ with $\alpha = 1.2$, $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left panel), $t = 1$ (middle panel), and $t = 2$ (right panel).

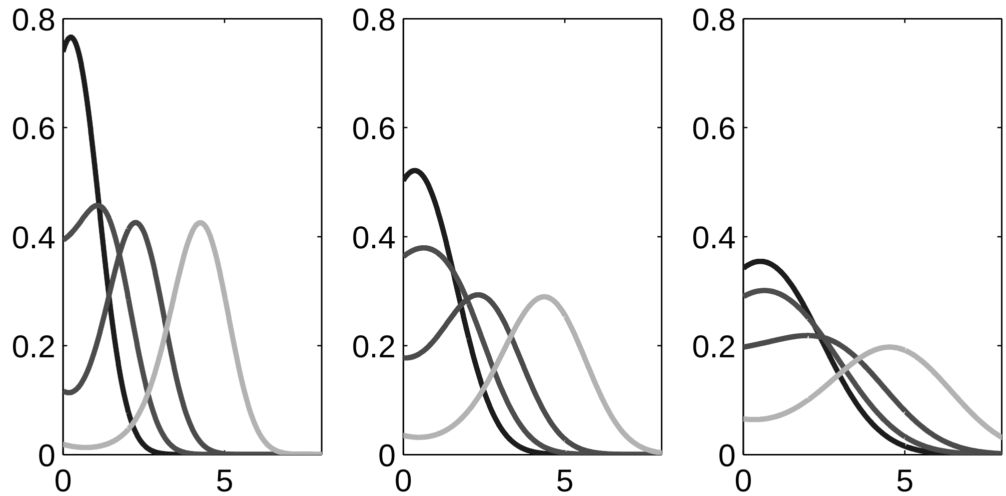

Figure 2. The transition densities of $p(x, y, t)$ with $\alpha = 1.8$, $x = 0, 1, 2, 4$ (left to right) at times $t = 0.5$ (left panel), $t = 1$ (middle panel), and $t = 2$ (right panel).

3] compute the backward generator of a killed stable Lévy process. It may be possible to develop the forward equation and compute the transition density for those process, using the methods of this paper. This would be interesting for applications to fractional diffusion, since it could elucidate the relevant fractional boundary conditions.

4. Fractional Cauchy problems

In this section, we show that the reflected stable process (1.2) with index $1 < \alpha \leqslant 2$ can be used as a time change to solve the fractional Cauchy problem

$$\partial_t^\beta p(x,t) = Lp(x,t); \quad p(x,0) = f(x) \tag{4.1}$$

of order $\beta = 1/\alpha$, when L generates a Feller process. The time-fractional Caputo derivative in (4.1) is defined by

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^\infty f'(t-r) r^{-\beta} dr, \tag{4.2}$$

a special case of (2.6). Fractional Cauchy problems are useful in a wide variety of practical applications [21, 39, 40, 47], and the next result allows a Markovian particle tracking solution for such problems [10, 54, 55].

Theorem 4.1. *For any $\beta \in [1/2, 1)$, let Z_t be given by (1.2), where Y_t is an α-stable Lévy process with characteristic function (1.1) for $\alpha = 1/\beta$. If X_t is an independent Markov process such that $T_t f(x) = \mathbb{E}^x[f(X_t)]$ forms a uniformly bounded, strongly continuous semigroup with generator L on some Banach space $\mathbb{B}$ of real valued functions, then $p(x,t) = \mathbb{E}^x[f(X_{Z_t})]$ solves the fractional Cauchy problem (4.1) for any $f \in \mathrm{D}(L)$, the domain of the generator.*

Proof. Theorem 3.1 in [4] states that if $u(x,t)$ solves the Cauchy problem $\partial_t u(x,t) = Lu(x,t)$; $u(x,0) = f(x)$ on $\mathbb{B}$ for some $f \in \mathrm{Dom}(L)$, then the solution to the associated fractional Cauchy problem (4.1) on $\mathbb{B}$ is given by

$$p(x,t) = \int_0^\infty u(x,r) h(r,t)\, dr \tag{4.3}$$

where

$$h(r,t) = \frac{t}{\beta} r^{-1-1/\beta}\, g_\beta(t r^{-1/\beta}) \tag{4.4}$$

and $g_\beta(t)$ is the stable probability density function with Laplace transform

$$\mathcal{L}g(s) = \int_0^\infty e^{-st} g_\beta(t)\, dt = e^{-s^\beta}$$

for some $0 < \beta < 1$. Since Y_t has no positive jumps, the first passage time $D_t := \inf\{r > 0 : Y_r > t\}$ is a stable subordinator with $\mathbb{E}[e^{-sD_t}] = \exp(-ts^{1/\alpha})$, and the supremum process $S_t = \sup\{Y_r : 0 \leqslant r \leqslant t\}$ is also the first passage time $E_t = \inf\{u > 0 : D_u > t\}$ of D_t, see Bingham [13]. Note that $\mathbb{P}(D_{E_t} > t) = 1$ [12, Theorem III.4]. Apply [33, Corollary 3.1] (or Exercises 29.7 and 29.18 in Sato [46]) to see that (4.4) is also the probability density of E_t. It follows from [12, Section VI.1, Prop. 3] that

$$\mathbb{P}(S_t \geqslant x) = \mathbb{P}(Z_t \geqslant x) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{4.5}$$

Hence (4.4) is also the probability density of Z_t, and the theorem follows. □

Remark 4.2. An alternate proof of Theorem 4.1 uses the reflection principle for spectrally negative stable Lévy processes. Extending the usual argument for reflected Brownian motion, let $\tau_x = \inf\{u > 0 : Y_u > x\}$. Since Y_t is self-similar, we have $\mathbb{P}(Y_t > 0) = 1/\alpha$ for every $t > 0$ (e.g., see [6, Theorem 4.1 (i)]). Defining $Y_t^x := Y_{t-\tau_x} - x$ for $t \geqslant \tau_x$, we have

$$\begin{aligned}\mathbb{P}(S_t \geqslant x) &= \mathbb{P}(S_t \geqslant x, Y_t > x) + \mathbb{P}(S_t \geqslant x, Y_t \leqslant x)\\ &= \mathbb{P}(Y_t > x) + \mathbb{P}(\tau_x \leqslant t, Y_t^x \leqslant 0)\end{aligned}$$

and since $\mathbb{P}(\tau_x \leqslant t, Y_t^x \leqslant 0) = (1-\alpha^{-1})\mathbb{P}(S_t \geqslant x)$ it follows that

$$\mathbb{P}(S_t \geqslant x) = \alpha\mathbb{P}(Y_t > x) = \mathbb{P}(Y_t > x | Y_t \geqslant 0) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{4.6}$$

An application of Zolotarev duality for stable densities [6, Theorem 4.1 (ii)] implies

$$\mathbb{P}(E_t > x) = \mathbb{P}(Y_t > x | Y_t \geqslant 0) \quad \text{for all } t > 0 \text{ and all } x > 0. \tag{4.7}$$

Then the theorem follows using (4.5), [4, Theorem 3.1] and [33, Corollary 3.1].

Remark 4.3. Theorem 4.1 confirms a conjecture in the paper [6, Remark 5.2]. There we set $Z_t = Y_{\sigma(t)}$ where $\sigma(t) = \inf\{u > 0 : H_u > t\}$ and $H_u = \int_0^u \mathbb{1}_{Y_s > 0}\, ds$. In essence, the negative excursions are cut away, and the positive excursions are joined together without any gaps in time. Since Y_t has no positive jumps, any up-crossing at the origin is a renewal point, so this process has the same distribution as (1.2).

5. Resolvent estimates

In this section, we develop bounds on the norm of the resolvent used in Theorem 2.1. Note that both terms in the formula (2.8) for the resolvent $R(\lambda, A)$ diverge as $x \to \infty$. For example, it follows directly from [22, Eq. (6.4)] that $E_\alpha(\lambda x^\alpha) \sim \alpha^{-1} e^{\lambda^{1/\alpha} x}$ as $x \to \infty$. Hence it is useful to begin by establishing an alternative representation. Recall that Y_t is a negatively skewed stable process with index $1 < \alpha \leqslant 2$ and characteristic function (1.1). Let $g_\alpha(x)$ denote the probability density function of $-Y_1$, a totally positively skewed stable law taking values on the entire real line.

Lemma 5.1. *For any $g \in C_\infty[0,\infty)$ and any $\operatorname{Re}\lambda > 0$ we have*

$$\begin{aligned}R(\lambda, A)g(x) = &\int_0^\infty \int_0^\infty e^{-\lambda t} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right) g(\xi)\, dt\, d\xi\\ &+ \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}} \int_0^\infty e^{-\lambda t} \frac{x}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt.\end{aligned} \tag{5.1}$$

where $\mathcal{L}g$ is the usual Laplace transform.

Proof. First we will show that the right-hand side of (5.1) vanishes for any $x < 0$. Recall that $g_\alpha(x)$ is a standard positively skewed stable density with index $1 < \alpha \leqslant 2$ and characteristic function $\exp((-ik)^\alpha)$, and hence $g_\alpha(-x)$ is a standard negatively skewed stable density with index $1 < \alpha \leqslant 2$ and characteristic function $\exp((ik)^\alpha)$, i.e., the density of Y_1 in (1.1). For $0 < \beta < 1$, let $g_\beta(x)$ denote the standard positively skewed stable density with characteristic function $\exp(-(-ik)^\beta)$. Since this density is supported on the positive real line, we can also write

$$\mathcal{L}g_\beta(s) = \exp(-s^\beta) \quad \text{for all } \operatorname{Re} s > 0, \tag{5.2}$$

(e.g., see [56, Lemma 2.2.1]). Apply the Zolotarev Duality Theorem for stable densities [56, Theorem 2.3.1] to see that when $\beta = 1/\alpha$ we have

$$g_\alpha(-x) = x^{-1-\alpha} g_\beta(x^{-\alpha}) \quad \text{for all } x > 0. \tag{5.3}$$

Then for $x < 0$, the first term in (5.1) is

$$\begin{aligned}
&\int_{-\infty}^{x}\int_0^\infty e^{-\lambda t}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{u}{t^{1/\alpha}}\right) g(x-u)\,dt\,du \\
&\quad= \int_{-\infty}^{x}\int_0^\infty e^{-\lambda t}\frac{1}{t^{1/\alpha}}\left(-\frac{u}{t^{1/\alpha}}\right)^{-1-\alpha} g_{1/\alpha}\left(\left(-\frac{u}{t^{1/\alpha}}\right)^{-\alpha}\right) g(x-u)\,dt\,du \\
&\quad= -\int_{-\infty}^{x}\int_0^\infty \frac{d}{d\lambda}\left[e^{-\lambda t}\right]\frac{1}{(-u)^{1+\alpha}} g_{1/\alpha}\left(\frac{t}{(-u)^\alpha}\right) dt\, g(x-u)\,du \\
&\quad= \int_{-\infty}^{x}\frac{1}{u}\frac{d}{d\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{(-u)^\alpha} g_{1/\alpha}\left(\frac{t}{(-u)^\alpha}\right) dt\, g(x-u)\,du.
\end{aligned}$$

Using (5.2) we have

$$\int_0^\infty e^{-\lambda t}\frac{1}{(-u)^\alpha} g_{1/\alpha}\left(\frac{t}{(-u)^\alpha}\right) dt = \exp(-[\lambda(-u)^\alpha]^\beta) = \exp(u\lambda^{1/\alpha}), \tag{5.4}$$

and then the first term in (5.1) equals

$$\begin{aligned}
\int_{-\infty}^{x}\frac{1}{u}\frac{d}{d\lambda}\left[e^{u\lambda^{1/\alpha}}\right] g(x-u)\,du =& \frac{1}{\alpha}\lambda^{1/\alpha-1}\int_0^\infty e^{(x-y)\lambda^{1/\alpha}} g(y)\,dy \\
=& \frac{1}{\alpha}\lambda^{1/\alpha-1} e^{x\lambda^{1/\alpha}}\mathcal{L}g(\lambda^{1/\alpha}) \\
=& \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\alpha\lambda^{1-1/\alpha}}\int_0^\infty e^{-\lambda t}\frac{1}{(-x)^\alpha} g_{1/\alpha}\left(\frac{t}{(-x)^\alpha}\right) dt \\
=& \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}}\int_0^\infty e^{-\lambda t}\frac{(-x)}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt
\end{aligned}$$

where we have used (5.4) again the next-to-last line. Since the last line above is the negative of the second term in (5.1), it follows that the sum of these two terms vanishes for all $x < 0$.

Since the positively skewed stable density $g_\alpha(x)$ tends to zero at a super-exponential rate as $x \to -\infty$ (e.g., see [56, Theorem 2.5.2]), its bilateral Laplace transform $\mathcal{L}_{-\infty}[g_\alpha](s)$ is well defined for all $s > 0$, and in fact we can write

$$\int_{-\infty}^\infty e^{-sx}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dx = e^{ts^\alpha} \quad \text{for all } t > 0 \text{ and all } s > 0. \tag{5.5}$$

Then it follows that

$$\int_0^\infty e^{-\lambda t}\int_{-\infty}^\infty e^{-sx}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dx\,dt = \frac{1}{\lambda - s^\alpha} \quad \text{for all } s > 0 \text{ and all } \operatorname{Re}\lambda > 0. \tag{5.6}$$

Then for any $g \in C_\infty[0,\infty)$ the convolution property of the bilateral Laplace transform implies that the first term in (5.1) satisfies

$$\int_{-\infty}^{\infty} e^{-sx} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\lambda t} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right) g(\xi)\, dt\, d\xi\, dx = \frac{\tilde{g}(s)}{\lambda - s^\alpha} \tag{5.7}$$

for all $s > 0$ and all $\operatorname{Re}\lambda > 0$.

As for the second term, in view of the fact that the complex contour integral

$$\int_\lambda^\infty \frac{d}{ds}\left[\frac{1}{u - s^\alpha}\right] du = \frac{\alpha s^{\alpha-1}}{\lambda - s^\alpha}$$

(integrate along $\{\lambda + r : r > 0\}$) it follows using (5.6) that

$$\begin{aligned}
\frac{s^{\alpha-1}}{\lambda - s^\alpha} &= \frac{1}{\alpha}\int_\lambda^\infty \frac{d}{ds}\left[\int_0^\infty e^{-ut}\int_{-\infty}^{\infty} e^{-sx}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dx\, dt\right] du\\
&= \int_{-\infty}^{\infty} e^{-sx}\int_0^\infty \left(\frac{-x}{\alpha}\right)\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right)\left[\int_\lambda^\infty e^{-ut}\, du\right] dt\, dx\\
&= -\int_{-\infty}^{\infty} e^{-sx}\int_0^\infty e^{-\lambda t}\left(\frac{x}{\alpha}\right)\frac{1}{t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt\, dx.
\end{aligned}$$

Then it follows immediately that the bilateral Laplace transform of the second term in (5.1) equals the second term in (2.9). Since the right-hand side of (5.1) vanishes for $x < 0$, its bilateral Laplace transform equals its ordinary Laplace transform. (Note however that neither term on the right-hand side of (5.1) vanishes for $x < 0$, only their sum.) But then (5.1) has the same Laplace transform as $R(\lambda, A)g(x)$, and since both are continuous in view of (2.8), the result follows using the uniqueness of the Laplace transform. □

Lemma 5.2. *Under the assumptions of Theorem 2.1, we have for every* $1 < \alpha \leqslant 2$ *that* (2.11) *holds for all* $\operatorname{Re}\lambda > 0$ *and all* $g \in C_\infty[0,\infty)$*, with* $M = M_\alpha + 1 + \sec(\pi/(2\alpha))$ *for some* M_α *depending only on* α*.*

Proof. Denote by $\mathrm{BUC}(\mathbb{R})$ the Banach space of bounded, uniformly continuous functions with the supremum norm. Given $g \in C_\infty[0,\infty)$, define $\bar{g} \in \mathrm{BUC}(\mathbb{R})$ by setting $\bar{g}(x) = g(x)$ for $x > 0$ and $\bar{g}(x) = g(0)$ for $x \leqslant 0$. Use (5.1) to write $R(\lambda, A)g(x) = I_1 - I_2 + I_3$ where

$$\begin{aligned}
I_1 &= \int_0^\infty \int_{-\infty}^{\infty} e^{-\lambda t}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right)\bar{g}(\xi)\, d\xi\, dt\\
I_2 &= \int_0^\infty \int_{-\infty}^{0} e^{-\lambda t}\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right)\bar{g}(\xi)\, d\xi\, dt\\
I_3 &= \frac{\mathcal{L}g(\lambda^{1/\alpha})}{\lambda^{1-1/\alpha}}\int_0^\infty e^{-\lambda t}\frac{x}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt.
\end{aligned} \tag{5.8}$$

The formula

$$T_t^\alpha f(x) = \int_{-\infty}^{\infty} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right) f(\xi)\,d\xi$$
$$= \int_{-\infty}^{\infty} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{\xi}{t^{1/\alpha}}\right) f(x-\xi)\,d\xi := \int_{-\infty}^{\infty} g_{t,\alpha}(\xi) f(x-\xi)\,d\xi, \quad f \in \mathrm{BUC}(\mathbb{R}),$$

defines a strongly continuous convolution semigroup on $\mathrm{BUC}(\mathbb{R})$. Indeed, $\bar{T}_t^\alpha f = f * g_{t,\alpha}$, $f \in L_1(\mathbb{R})$, defines a strongly continuous semigroup on $L_1(\mathbb{R})$, see, e.g. [24, Theorem 21.4.3], noting that the Fourier transform of $g_{t,\alpha}$ is $e^{t(ik)^\alpha}$. The latter follows from (5.5) which also holds for $s = ik$, as g_α is absolutely integrable. Then, T_t^α is a subordinate semigroup (where the right-translation group on $\mathrm{BUC}(\mathbb{R})$, which is strongly continuous [19, Chapter I, Section 4.15], is subordinated against $\bar{T}_{\alpha,t}$) which is strongly continuous by [5, Theorem 4.1].

Next we show that T_t^α is a bounded analytic semigroup on $\mathrm{BUC}(\mathbb{R})$, by showing that $\|\frac{d}{dt} T_t^\alpha f\| = \|A_\alpha T_t^\alpha f\| \leqslant M t^{-1}\|f\|$ for some $M > 0$, see [3, Theorem 3.7.19]. Here, the operator A_α denotes the generator of T_t^α. We have that

$$\left\|\frac{d}{dt} T_t^\alpha f\right\| \leqslant \int_{-\infty}^{\infty} \left|\frac{d}{dt} g_{t,\alpha}(x)\right| dx\,\|f\|.$$

Recall Carlson's inequality (see, [15, 18])

$$\int_{-\infty}^{\infty} |h(x)|\,dx \leqslant C\left(\int_{-\infty}^{\infty} |\mathcal{F}[h](k)|^2\,dk\right)^{1/4}\left(\int_{-\infty}^{\infty}\left|\frac{d}{dk}\mathcal{F}[h](k)\right|^2 dk\right)^{1/4}$$

where $\mathcal{F}[h](k) = \int e^{-ikx} h(x)\,dx$ denotes the Fourier transform of h. Note that $\mathcal{F}[\frac{d}{dt} g_{t,\alpha}](k) = (ik)^\alpha e^{t(ik)^\alpha}$. It is easy to check, using the formula for the gamma probability density, that

$$\int_{-\infty}^{\infty} \left|(ik)^\alpha e^{t(ik)^\alpha}\right|^2 dk \leqslant \int_{-\infty}^{\infty} |k|^{2\alpha} e^{-2tc_\alpha |k|^\alpha} dk \leqslant C_\alpha t^{-2-\frac{1}{\alpha}}$$

for some $c_\alpha, C_\alpha > 0$, and that

$$\int_{-\infty}^{\infty} \left|\frac{d}{dk}\left((-ik)^\alpha e^{t(-ik)^\alpha}\right)\right|^2 dk \leqslant 2\int_{-\infty}^{\infty} \left(|k|^{2(\alpha-1)} + t^2|k|^{2(2\alpha-1)}\right) e^{-2c_\alpha t|k|^\alpha} dk \leqslant C_\alpha t^{-2+\frac{1}{\alpha}}.$$

Hence $\|A_\alpha T_t^\alpha f\| \leqslant K_\alpha t^{-1}\|f\|$, and so T_t^α is a bounded analytic semigroup on $\mathrm{BUC}(\mathbb{R})$. Therefore, by Corollary [3, Corollary 3.7.12],

$$\|\lambda I_1\| = \left\|\lambda \int_0^\infty e^{-\lambda t} T_t^\alpha \bar{g}\,dt\right\| = \|\lambda R(\lambda, A_\alpha)\bar{g}\| \leqslant M_\alpha \|\bar{g}\| = M_\alpha \|g\|, \quad \operatorname{Re}\lambda > 0.$$

Next write

$$
\begin{aligned}
|\lambda I_2| \leqslant & |\lambda g(0)| \left\| \int_0^\infty \int_{-\infty}^0 e^{-\lambda t} \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x-\xi}{t^{1/\alpha}}\right) d\xi \, dt \right\| \\
\leqslant & |\lambda| \|g\| \left\| \int_0^\infty e^{-\lambda t} \int_{\frac{x}{t^{1/\alpha}}}^\infty g_\alpha(y) \, dy \, dt \right\| \qquad (5.9) \\
= & |\lambda| \|g\| \left\| - \frac{e^{-\lambda t}}{\lambda} \int_{\frac{x}{t^{1/\alpha}}}^\infty g_\alpha(y) \, dy \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} \frac{x}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt \right\|
\end{aligned}
$$

A substitution $t = (x/u)^\alpha$ leads to

$$
\int_0^\infty e^{-\lambda t} \frac{x}{\alpha t^{1+1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt = \int_0^\infty e^{-\lambda (x/u)^\alpha} g_\alpha(u) \, du, \tag{5.10}
$$

and then it follows that

$$
|\lambda I_2| \leqslant \|g\| \left\| \int_0^\infty e^{-\lambda \left(\frac{x}{y}\right)^\alpha} g_\alpha(y) \, dy \right\| \leqslant \|g\|
$$

for all $\operatorname{Re} \lambda > 0$ and all $g \in C_\infty[0, \infty)$.

Another application of (5.10) shows that $|\lambda I_3| \leqslant |\lambda^{1/\alpha} \mathcal{L}g(\lambda^{1/\alpha})|$. If $\lambda = re^{i\theta}$ for some $r > 0$ and $|\theta| < \pi/2$ then $\lambda^{1/\alpha} = r^{1/\alpha} e^{i\theta/\alpha}$ has real part $r^{1/\alpha} \cos(\theta/\alpha) \geqslant r^{1/\alpha} \cos(\pi/(2\alpha))$. Hence $|\lambda^{1/\alpha}| = r^{1/\alpha} \leqslant \operatorname{Re}[\lambda^{1/\alpha}] / \cos(\pi/(2\alpha))$ for all $\operatorname{Re} \lambda > 0$. It is not hard to check that $\operatorname{Re}[\lambda |\mathcal{L}g(\lambda)|] \leqslant \|g\|$ for all $\operatorname{Re} \lambda > 0$ and all $g \in C_\infty[0, \infty)$. Then we have

$$
|\lambda I_3| \leqslant |\lambda^{1/\alpha} \mathcal{L}g(\lambda^{1/\alpha})| = \left| \frac{|\lambda^{1/\alpha}|}{\operatorname{Re}[\lambda^{1/\alpha}]} \cdot \operatorname{Re}[\lambda^{1/\alpha} |\mathcal{L}g(\lambda^{1/\alpha})|] \right| \leqslant \frac{1}{\cos(\pi/(2\alpha))} \|g\|
$$

for all $\operatorname{Re} \lambda > 0$ and all $g \in C_\infty[0, \infty)$, and the result follows. □

Remark 5.3. Following a slightly different path using Fourier transforms, it is possible to show that

$$
\lambda R(\lambda, A) g(x) = \lambda R(\lambda, D_x^\alpha) \bar{g}(x) - \lambda R(\lambda, D_x^\alpha) g(0) \mathbb{1}_{(-\infty,0)}(x) + c_\alpha (\lambda R(\lambda, D_x^\alpha) - \mathrm{I}) \mathbb{1}_{(-\infty,0)}(x)
$$

where $\bar{g}(x) = g(x)$ for $x > 0$, $\bar{g}(x) = g(0)$ for $x \leqslant 0$, $c_\alpha = \lambda^{1/\alpha} \mathcal{L}g(\lambda^{1/\alpha})$, I is the identity operator, and $\mathbb{1}_{(-\infty,0)}(x)$ is the indicator function. This form clarifies that the resolvent of the Caputo fractional derivative $A = \partial_x^\alpha$ is a modification of the resolvent of the Riemann-Liouville fractional derivative D_x^α to account for the boundary term, which is natural in view of (2.7). The same resolvent bound in Lemma 5.2 can also be obtained using this form.

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Appendix

The following Matlab code computes the transition density $p(x, y, t)$ in the case $x = 2$ for the reflected stable process Z_t defined by (1.2), where Y_t is a stable Lévy process with characteristic function (1.1) and index $1 < \alpha < 2$. This code was used to generate the plots in Figures 1 and 2.

```
%%% Matlab script to compute p(x,y,t)
%% enter variables
    alpha=1.2; ymax=12; N=1200; t=[0,.5,1,2]; x=2;
%% initialise parameters
    h=ymax/N; y=(h:h:ymax)';
    u0=zeros(N,1);u0(floor(x/h)+1)=1/h; % initial condition
%% Make Grunwald matrix
    w=ones(1,N+1);
    for k=1:N
      w(k+1)=w(k)*(k-alpha-1)/k;
    end
    w=w/h^alpha;
    M=spdiags(repmat(w,N,1),-1:1:N-1,N,N); %enter w's along diagonals
    M(1,:)=-cumsum(w(1:N))'; %change first row for BC
%% Solve ODE system
    [~,p]=ode113(@(t,u) M*u,t,u0);
```

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Boris Baeumer, Department of Mathematics and Statistics, University of Otago, Dunedin, NZ
E-mail address: bbaeumer@maths.otago.ac.nz

Mihály Kovács, Department of Mathematics and Statistics, University of Otago, Dunedin, NZ
E-mail address: mkovacs@maths.otago.ac.nz

Mark M. Meerschaert, Department of Probability and Statistics, Michigan State University
E-mail address: mcubed@stt.msu.edu
URL: http://www.stt.msu.edu/users/mcubed/

René L. Schilling, Institut für Mathematische Stochastik, Technische Universität Dresden, Germany
E-mail address: rene.schilling@tu-dresden.de
URL: http://www.math.tu-dresden.de/sto/schilling/

Peter Straka, School of Mathematics and Statistics, The University of New South Wales, Australia
E-mail address: p.straka@unsw.edu.au