Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2022

Questions for Part 2

Instructions to Candidates

You have a maximum of **fifty minutes** to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

DEFINITION

A *prime number* has exactly two factors, 1 and itself. By this definition, 1 is not a prime number.

The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

In this question you do not need to show working, except in part (c).

(a) Find the value of each of the following. Give your answers as fractions; answers in decimal form will earn no marks.

(i) 3/4 of 7.

(ii) 1/2 ÷ 2/5.

(iii) 2/7 − 1/5.

(b) Consider the sum of two fractions $a/b + c/d$, where a, b, c, and d are integers.

(i) Write the sum of two fractions $a/b + c/d$ as a simple fraction.

(ii) Under what circumstances is the sum of two fractions $a/b + c/d$ not defined?

(c) Mi-rae uses 1/5 of her weekly baby-sitting money to buy a birthday present for her brother and puts 1/3 of the remaining money into a savings account. If she has $32 left, how much did she have at first?

Question 2: 10 marks (Years 10 and below only)

Farmer Gerard decides to make three rectangular pens for his sheep.

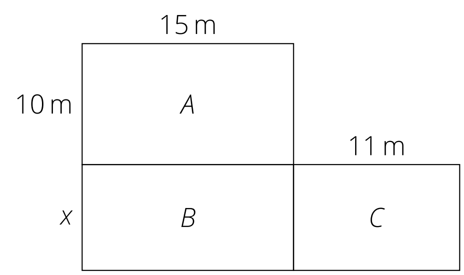
(a) Find the area of the pen labelled A.

(b) Write as an expression the area of the pen labelled B.

(c) Write as an expression the perimeter of the pen labelled C.

(d) Write as an expression the total exterior perimeter of all three pens combined.

(e) Suppose the total exterior perimeter of all three pens combined is 90 m. What is the value of x?



(NOT TO SCALE)

Question 3: 20 marks (All Years)

Note: In this question when we write abc, we mean $abc = 100 \times a + 10 \times b + c$, where a, b, and c are single digit numbers. A similar situation applies for ab, $abcd$, and $abcde$. Also, if z is an integer, then a *perfect square* is a number of the form $z \times z$ (also written as z^2), while a *cube* is a number of the form $z \times z \times z$.

(a) Find a three digit number abc (with $a \neq 0$ and $c \neq 0$) such that abc is an odd perfect square and cba is even.

(b) Find a four digit number $abcd$ (with $a \neq 0$ and $d \neq 0$) such that $abcd$ is a cube whose digits add to a prime number.

(c) Find a five digit number $abcde$ (with a, b, c, d, and e *all* being nonzero) such that $abcde = edcba$ and the digits of $abcde$ add to an even multiple of 9.

(d) Find a three digit number abc (with $a \neq 0$ and $c \neq 0$) such that abc is a perfect square and cba is a prime number. (You don't need to show cba is prime for full marks.)

(e) Find all numbers of the form ab (with $a \neq 0$ and $b \neq 0$) where ab and ba are *both* prime numbers. (For example, 23 is not a number of this type as 32 is even.)

Question 4: 20 marks (All Years)

Suppose for a given number we square each of its digits and sum these squares. Further suppose we repeat this process (noting each sum down in turn) until we either reach a sum of 1 or we reach a sum that we have already noted down.

If we reach a sum of 1, we call our original number a *happy number*. Otherwise our original number is a *sad number*. For example, 13 is a happy number: $1^2 + 3^2 = 10$, and $1^2 + 0^2 = 1$. An example of a sad number is 4 (including 4, our list of sums in this case is 4, 16, 37, 58, 89, 145, 42, 20, and back to 4; as such, all the numbers in this list are sad numbers).

(a) What is the smallest possible happy number?

(b) Use the fact that 4 is a sad number to show that 11 is also a sad number.

(c) Is 19 a happy number or a sad number? Show your working (list the sums you must note down to determine whether 19 is happy or sad).

(d) How many 2 digit numbers starting with 1 are happy numbers?

A *Harshad number* is a number that is divisible by the sum of its digits. For example, 12 is a Harshad number since $12/(1+2) = 4$. A number is called *twice-Harshad* if the number we get when we divide our original number by the sum of its digits is also Harshad.

(e) Are all single digit numbers greater than zero Harshad? If so, briefly explain why this is, and if not, give an example of a single digit number greater than zero that is not Harshad.

(f) Which numbers out of 13, 24, 46, 100, 343, 3100 are not Harshad?

(g) Which numbers out of 13, 24, 46, 100, 343, 3100 are twice Harshad?

Question 5: 20 marks (All Years)

(a) Find the non-even prime factor of 272.

(b) Triangular numbers take the form $1/2n(n+1)$ for some integer n greater than 0. Using this formula, find the 272nd triangular number.

(c) Show that 272 is a *pronic number* — that is, it is the product of two consecutive integers, or twice a triangular number.

(d) Show that 272 is the sum of four consecutive prime numbers. (Hint: first divide 272 by 4.)

(e) Suppose you have constructed an isosceles triangle with a base side of 272 units. Let h be the height of this triangle, a be the area of this triangle, and s be the length of each of this triangle's other two sides. If h, a, and s are all positive integers, find one set of possible values for h, a, and s. (Hint: remember that the area of a triangle is half its base times its height.)

Question 6: 20 marks (All Years)

Suppose we have a freight container whose internal dimensions are 2.5 metres wide by 12 metres long by 3 metres high. Into this container may be placed boxes of different sizes.

(a) Consider a metal box with dimensions 200 mm by 200 mm by 200 mm. What is the maximum number of these boxes that we could fit into our freight container?

(b) Consider another metal box with dimensions 200 mm by 480 mm by 480 mm. If each box has the same orientation inside the container, what is the maximum number of these boxes that we could fit into our freight container? (Hint: the box can have three different orientations.)

(c) Consider a third metal box with dimensions 200 mm by 480 mm by 625 mm. If each box has the same orientation inside the container, what is the maximum number of these boxes that we could fit into our freight container?

Question 7: 10 marks (Years 10 and 11 only)

While walking along the beach one day Kitty finds a rather strange treasure map inside a bottle. Each direction on the map is given as a mathematical puzzle.

The directions are as follows:

(a) Starting at the rock shaped like a dragon's head at the north end of the beach, face due east, then rotate 30° anti-clockwise. Walk 5 metres in the direction you are facing.

(b) From the point reached in (a), walk $f(6)$ metres north, then $f(7)$ metres east, then $f(8)$ metres north, and finally $f(9)$ metres east, where $f(n)$ gives you the nth Fibonacci number (which is the sum of the previous two Fibonacci numbers; the first 5 numbers in this sequence are $f(1) = 0$, $f(2) = 1$, $f(3) = 1$, $f(4) = 2$, and $f(5) = 3$).

(c) Solve the simultaneous equations $3x + 2y = 22$ and $7x - 3y = 36$, then walk x metres west and y metres south from the point you reached in (b). Start digging!

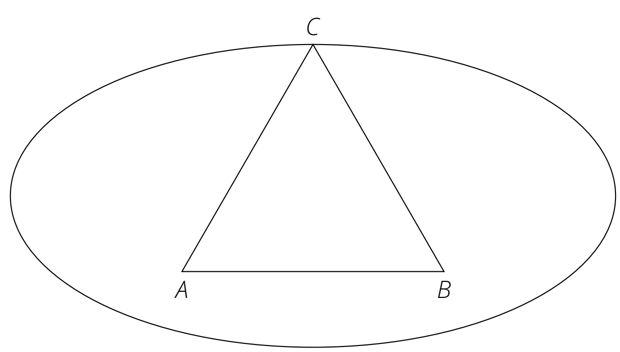
Being rather clever Kitty finds the treasure in no time at all. Follow the instructions and determine how many metres east and how many metres north the treasure is from Kitty's starting point at the rock. Round your answers to 3 significant figures.

Question 8: 10 marks (Year 11 only)

Note: the formula for the area of the ellipse is $ab\pi$, where a is half the height of the ellipse, and b is half the width of the ellipse.

Suppose we have an ellipse twice as wide as it is tall. An equilateral triangle is drawn inside the ellipse, such that:

- The centre of the triangle is the centre of the ellipse.
- One of the corners of the triangle (C) co-incides with the ellipse's highest point.
- One of the sides ($\overline{AB}$) of the triangle is parallel to the tangent to the ellipse at C.



If the area of the ellipse is between 40 and 50 square centimetres and the height of the triangle is a whole number of centimetres, find the area of the ellipse to two decimal places.

(END OF COMPETITION)