Department of Mathematics and Statistics
UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2024

Questions for Part 2

Instructions to Candidates

You have a maximum of **fifty minutes** to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

Note: There are **four** pages in this question booklet: this instruction page and **three** pages of questions.

DEFINITION

A *prime number* has exactly two factors, 1 and itself. By this definition, 1 is not a prime number.

The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

Joanna has received a \$250 000 inheritance from her rich grandmother. She plans to invest some of it, and she will spend the other 12.5% of the money.

(a) Write down 12.5% as a decimal.

(b) Using the number you wrote down in (a), in one line write down the working needed to show that the amount she will spend is \$31 250.

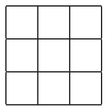
(c) Joanna decides to use 3/20 of the spending money on a holiday. How much will she spend on the holiday?

(d) For her holiday, Joanna allocates \$750 for travel expenses. If she spends 14 nights of her on holiday at an hotel costing \$150 a night, how much of her holiday money will she have after accounting for her travel and accommodation costs?

(e) What fraction of the original \$250 000 Joanna inherited is she spending on the holiday? Give your answer in its simplest fraction form.

Question 2: 10 marks (Years 10 and below only)

First **copy** the square array shown below into your answer book, then find nine consecutive natural numbers and fill them into the square array in your answer book, such that the sum of the three numbers in each row, column, and diagonal of the array is equal to 60.



Question 3: 20 marks (All Years)

2024 can be written as $2^3 \times 11 \times 23$. Put another way, 2024 is of the form p^3qr, where p, q, and r are distinct prime numbers and $r = 2q + 1$. (This definition of p, q, and r is used throughout (a), (b), (c), and (d).)

(a) Find the smallest number of the form p^3qr.

(b) If $p = 2$ find the smallest number of the form p^3qr larger than 2024.

(c) Is the number you found in (b) the smallest number of the form p^3qr larger than 2024 (allowing for $p \neq 2$)? If it is, briefly explain why there is no number of the form p^3qr larger than 2024 but smaller than your answer for (b). If it isn't, give another number of the form p^3qr larger than 2024 but smaller than the number you found in (b).

(d) Find all numbers of the form p^3qr smaller than 2024. You do not need to include the number you listed in (a).

2024 is the 39th time the Junior Mathematics Competition has run. 39 is the sum of 5 consecutive prime numbers:

$$3 + 5 + 7 + 11 + 13 = 39.$$

(e) Find all other numbers smaller than 100 that are the sum of 5 consecutive prime numbers.

(f) Which of the numbers you found in (e) are also prime numbers?

Question 4: 20 marks (All Years)

Books published before 2007 have a string of 10 digits on them called an *International Standard Book Number* or *ISBN*, that identifies the book. Each of the first 9 digits in the string is a number between 0 and 9 inclusive, and the 10th digit is either a number between 0 and 9 inclusive or an X.

In order for such a string to be an ISBN, it must satisfy an additional condition. Let x_1 denote the first digit, x_2 the second digit, etc. The last digit x_{10} must be the remainder on dividing

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 8x_8 + 9x_9$$

by 11. (If the remainder is 10, then x_{10} is X.)

(a) Briefly explaining your answer, is the string 038696617X an ISBN?

(b) The 8th digit of the ISBN printed on a book is illegible. The ISBN reads 0321149Z20, where the ‘Z’ is unknown. Find Z.

(c) A common error made when inputting numerical data is a “transposition” error, where two adjacent digits get swapped, e.g. someone trying to type “68” might accidentally type “86” instead. Show that if the 4th and 5th digits (which are different) are transposed when typing out an ISBN, the resulting string of digits is not an ISBN.

Question 5: 20 marks (All Years)

12 points are marked around a circle. A *k*-star is drawn as follows:

> *(1)* Choose any point.
>
> *(2)* From your current point, draw a straight line from that point to the *k*th point coming after (clockwise).
>
> *(3)* If the new point doesn't have a line already coming from it, repeat step (2), otherwise move to the next step.
>
> *(4)* If all points are incident to two lines, the *k*-star is complete. Otherwise choose a point without any lines going to it and repeat the process from step (2).

Note that when drawing the *k*-star we never draw a line between any two points more than once.

(a) Draw the 4-star and the 5-star.

(b) Can the same star be obtained for multiple values of *k*? Explain your answer.

(c) Which *k*-stars for *k* between 1 and 11 inclusive can be drawn without ever taking your pen off the paper?

(d) Now suppose 30 points are marked around a circle. Which *k*-stars for *k* between 1 and 29 inclusive can be drawn without ever taking your pen off the paper?

Question 6: 20 marks (All Years)

While doing a computing course, Carol investigates *8-bit* numbers. These are numbers written in *binary notation*; that is, every digit is written as either a 1 or a 0. 8-bit numbers have 8 digits in total, and always include the leading zeroes. For example, 00011001, 01111100, and 11100011 are all 8-bit numbers. (We never write our first two examples as 11001 or 1111100.)

If Carol writes an 8-bit number as *abcdefgh*, where a, b, c, d, e, f, g, and h are either 0 or 1, she can convert her number into decimal notation by using the following equation:

$$\text{(number in decimal notation)} = 2^7a + 2^6b + 2^5c + 2^4d + 2^3e + 2^2f + 2^1g + 2^0h.$$

Note that every 8-bit number has exactly one decimal equivalent, and vice versa.

(a) What is the decimal equivalent of the 8-bit number 11100100?

(b) What is the 8-bit equivalent of the decimal number 104?

(c) What is the decimal equivalent of the largest 8-bit number?

One way of combining two 8-bit numbers is to use a *logic gate*, where each digit of the two numbers is combined using a particular rule to produce either a 0 or 1 in the new 8-bit number. Here are three such rules:

> *(1)* AND: if both digits are 1, the result is 1, otherwise 0.
>
> *(2)* OR: if both digits are 0, the result is 0, otherwise 1.
>
> *(3)* XOR: if exactly one of the two digits is 1, the result is 1, otherwise 0.

For example, 01010111 AND 11100000 = 01000000 and 01010111 OR 11100000 = 11110111. The order we in which have our two numbers is not important: for example, 01010111 XOR 11100000 and 11100000 XOR 01010111 are both equal to 10110111.

The output of combining two 8-bit numbers with a logic gate is always another 8-bit number. This means that logic gates can be used multiple times to combine any number of 8-bit numbers into a single 8-bit number. Again the order of the inputs does not matter. For example, given four 8-bit numbers a, b, c, and d, the following combinations (**amongst others**) produce the same 8-bit number as a final output: (a OR b) AND (c OR d), (b OR a) AND (c OR d), (c OR d) AND (a OR B), and (d OR c) AND (b OR a).

(Question 6 continued on next page)

(Question 6 continues)

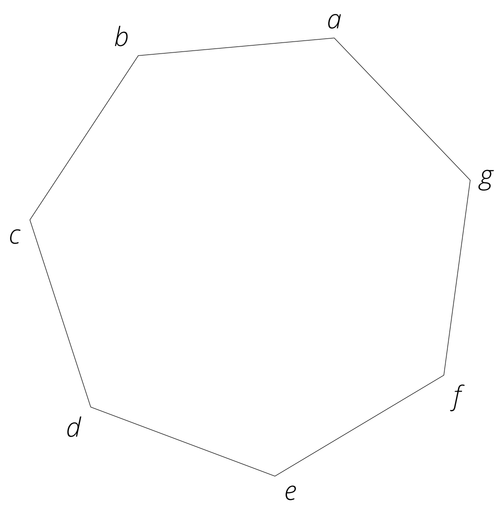
Suppose Carol wishes to combine the 8-bit numbers 11001100, 10101010, 11100110, and 00001001 into one 8-bit number using a combination of logic gates.

(d) Carol first combines 11001100 and 10101010 with AND, then 11100110 and 00001001 with OR. If she combines the outputs of these two logic gates with XOR, what final output does she get?

(e) If the output Carol wants is 01100110, provide a way she can join together any three of the logic gates described above with her four input numbers listed above to achieve this. (Each gate can be used multiple times or not at all, but exactly three gates are to be used.)

(f) Is the way you found in (e) the only way Carol could have reached 01100110 from her four chosen inputs? If so, briefly explain why this would be the case. If not, provide another way of joining together the logic gates.

Question 7: 10 marks (Years 10 and 11 only)



Consider a regular heptagon with side length 1 cm and with vertices labelled a, b, c, d, e, f, and g. Let T be the set of all triangles that can be constructed out of any three vertices of said heptagon, and let A be the subset of T containing every member of T with a as a vertex.

(a) How many triangles are in A?

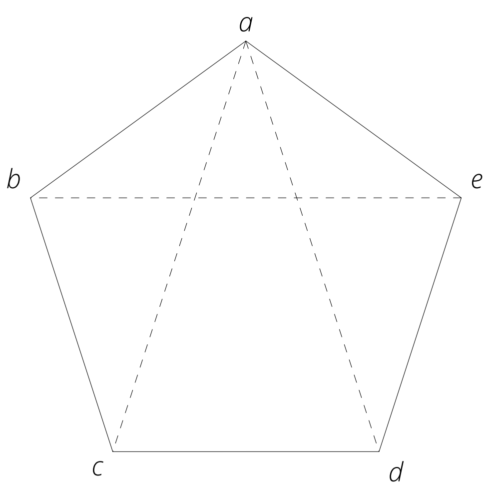
(b) How many triangles are in T?

We say two triangles s and t are *congruent* if for each side of s there is a side of t with the same length (and vice versa), and for each angle of s there is an angle of t of the same size (and vice versa). If s and t are congruent triangles we can denote this in shorthand by saying $s \cong t$ (if s and t are not congruent we write $s \ncong t$ instead).

(c) How many triangles in A are not congruent to $\triangle agf$? (Recall that all members of A have a as a vertex.)

(d) Suppose M is any subset of A where for any pair of triangles $s, t \in M$ we have $s \ncong t$. What is the most number of triangles that can be in M?

Question 8: 10 marks (Year 11 only)



Consider a regular pentagon with side length 1 cm and with vertices labelled a, b, c, d, and e. (The pictured diagram is *not to scale*.)

(a) Find the area of the triangle $\triangle abe$.

(b) Find the area of the triangle $\triangle acd$.

(END OF COMPETITION)