Department of Mathematics and Statistics

University of Otago
Te Whare Wānanga o Otāgo
New Zealand

# Junior Mathematics Competition 2022

## Student Report

### Year 9 (Form 3) Prize Winners

| | |
|---|---|
| **First** | Gwang Ho Kim, Rangitoto College |
| **Second** | Junyi Guo, Kristin School |
| **Third** | Wesley Lau, Macleans College |

**Top 30 (in School Order):**

Max McCulloch, Avondale College
Ericsson Ye, Christ's College
Enzo Li, Long Bay College
Jason Li, Macleans College
Lawrence Wen, Macleans College
Sophia Wang, Pinehurst School
Yanchen Tan, Rangitoto College
Matthew Bluck, St Andrew's College
Chen Sun, St Kentigern College
Rudra Patel, Avondale College
Xuanwei Chen, Diocesan School for Girls
Jessica Bao, Macleans College
Nichole Luo, Macleans College
Zehba Husein, Papanui High School
Devin Xu, Pinehurst School
Joshua Wang, Rangitoto College
Jackie Xu, St Cuthbert's College
Ethan Peng, St Paul's Collegiate School
Soomin Song, Burnside High School
Frank Deng, Kristin School
Cherry Lee, Macleans College
Jerry Qiu, Macleans College
Jack Chen, Pinehurst School
Tiantian Chen, Rangi Ruru Girls' School
Frank Jin, Scots College
Taylor Bain, St Joseph's School (Ashburton) / St Joseph's School (Papanui)
Hamish Bell, St Peter's College (Epsom)

### Year 10 (Form 4) Prize Winners

| | |
|---|---|
| **First** | Daniel Xian, St Kentigern College |
| **Second** | Josephine Sim, Macleans College |
| **Third** | Leon Lee, St Peter's School (Cambridge) |

**Top 30 (in School Order):**

Chenghao Li, ACG Parnell College
Alexander Lau, Aidanfield Christian School
Aditya Patel, Avondale College
Justin Cui, Macleans College
Chenyou Song, Macleans College
Tony Leo, Rangitoto College
Leo Wang, Rangitoto College
Yi Nan Chen, St Kentigern College
Eise Tijsen, Waimea College
Allen Weng, ACG Parnell College
Fengfeng Chen, Avondale College
Wesley Ee Wen Teh, Botany Downs Secondary College
Sunny Liu, Macleans College
Victor Coen, Mount Albert Grammar School
Selina Ni, Rangitoto College
Tony Yu, Scots College
Oscar Prestidge, St Kentigern College
Theo Keith, Wellington High School
Eric Mironov, ACG Sunderland
Rohan Kumar, Avondale College
Jessica Rankin, Burnside High School
Cecilia Ma, Macleans College
Ava Poynter, Mount Albert Grammar School
Chloe Seo, Rangitoto College
Alan Chen, St Kentigern College
Jasper Carran, St Peter's College (Epsom)
Elaine Zhou, Westlake Girls' High School

### Year 11 (Form 5) Prize Winners

| | |
|---|---|
| **First** | Nico McKinlay, St Kentigern College |
| **Second** | Joe Chan, Botany Downs Secondary College |
| **Third** | Andrew Chen, Macleans College |

**Top 30 (in School Order):**

Kaicheng Yuan, ACG Parnell College
Yash Naicker, Burnside High School
David Li, Kings College
Justin Huang, Macleans College
Raymond Zhang, Macleans College
Allen Li, Rangitoto College
Jifei Shao, St Cuthbert's College
Jonathan Chia, St Kentigern College
Devin Shen, Westlake Boys' High School
Seivin Kim, Avondale College
David Evans, Canterbury Home Educators
Winston Weng, Kings College
Yixue Wang, Macleans College
Christopher Mara, Mount Roskill Grammar School
Yeonsu Na, Rangitoto College
Belle Yin, St Cuthbert's College
Daniel Qin, St Kentigern College
Ubeen Sim, Westlake Boys' High School
Jingyuan Cui, Botany Downs Secondary College
Connor Gray, King's High School
Nicole Wong, Kristin School
Jesse Zhang, Macleans College
Luka Dunwoodie, Naenae College
Bryan Cooper, St Andrew's College
Ena Yin, St Cuthbert's College
Alan Bailey, Westlake Boys' High School
Yaoyu Xie, Westlake Boys' High School

## 2022 Model Solutions

As always only one method is shown. Often several methods exist. We don't guarantee that any method shown is the best or fastest in any case.

### Question 1: 10 marks (Years 9 and below only)

In this question you do not need to show working, except in part (c).

(a) Find the value of each of the following. Give your answers as fractions; answers in decimal form will earn no marks.

(i) 3/4 of 7.

■ $\frac{21}{4}$

(ii) $1/2 \div 2/5$.

■ $\frac{5}{4}$

(iii) $2/7 - 1/5$.

■ $\frac{3}{35}$

(b) Consider the sum of two fractions $a/b + c/d$, where $a$, $b$, $c$, and $d$ are integers.

(i) Write the sum of two fractions $a/b + c/d$ as a simple fraction.

■ $\frac{ad+cb}{bd}$

(ii) Under what circumstances is the sum of two fractions $a/b + c/d$ not defined?

■ If either $b = 0$ or $d = 0$ then $a/b + c/d$ is not defined (since then $bd = 0$ and fractions always have a non-zero denominator).

(c) Mi-rae uses 1/5 of her weekly baby-sitting money to buy a birthday present for her brother and puts 1/3 of the remaining money into a savings account. If she has \$32 left, how much did she have at first?

■ Let the amount of money she had be $x$. Then

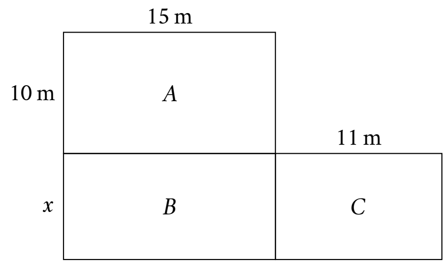
$$\begin{aligned} 32 &= 2/3(4/5x) \\ 32 &= 8/15x \\ x &= 32 \times 15/8 \\ x &= 60 \end{aligned}$$

She originally had \$60.

***In general students found most of this question straightforward, although the average mark was slightly lower than that for last year's competition. The most difficulty arose in (b), where a basic understanding of algebra is required. Like in previous years, when asked for a general solution too many students gave an example in the first part of (b) instead.***

### Question 2: 10 marks (Years 10 and below only)

Farmer Gerard decides to make three rectangular pens for his sheep.




(NOT TO SCALE)

(a) Find the area of the pen labelled $A$.

■ $10 \times 15 = 150\,\text{m}^2$.

(b) Write as an expression the area of the pen labelled $B$.

■ $15x$ metres or equivalent.

(c) Write as an expression the perimeter of the pen labelled $C$.

■ $22 + 2x$ metres or equivalent.

(d) Write as an expression the total exterior perimeter of all three pens combined.

■ The perimeters of each area are $30 + 20 = 50$ metres, $30 + 2x$ metres, and $22 + 2x$ metres respectively. When we combine these we must remove two lots of $x$ and two lots of 15 as these are shared by two areas. Thus the exterior perimeter is $72 + 2x$ metres.

(e) Suppose the total exterior perimeter of all three pens combined is 90 m. What is the value of $x$?

■ Using the expression found in (d) we have $72 + 2x = 90$, and thus $x = 9$.

***This was probably the easiest question in the competition overall, with the average mark similar to that of the equivalent question last year. The most common error seen by markers this year was in (c), where students often gave the area of the pen labelled* C *rather than its perimeter. In many cases students did not simplify their answer to (d).***

### Question 3: 20 marks (All Years)

Note: In this question when we write $abc$, we mean $abc = 100 \times a + 10 \times b + c$, where $a$, $b$, and $c$ are single digit numbers. A similar situation applies for $ab$, $abcd$, and $abcde$. Also, if $z$ is an integer, then a *perfect square* is a number of the form $z \times z$ (also written as $z^2$), while a *cube* is a number of the form $z \times z \times z$.

(a) Find a three digit number $abc$ (with $a \neq 0$ and $c \neq 0$) such that $abc$ is an odd perfect square and $cba$ is even.

■ Any of the following will work: $15 \times 15 = 225 \to 522$, $17 \times 17 = 289 \to 982$, $21 \times 21 = 441 \to 144$, $25 \times 25 = 625 \to 526$, and $29 \times 29 = 841 \to 148$.

(b) Find a four digit number $abcd$ (with $a \neq 0$ and $d \neq 0$) such that $abcd$ is a cube whose digits add to a prime number.

■ Any of the following will work: $13 \times 13 \times 13 = 2197 \to 2 + 1 + 9 + 7 = 19$, $14 \times 14 \times 14 = 2744 \to 2 + 7 + 4 + 4 = 17$, $16 \times 16 \times 16 = 4096 \to 4 + 0 + 9 + 6 = 19$, and $17 \times 17 \times 17 = 4913 \to 4 + 9 + 1 + 3 = 17$.

(c) Find a five digit number $abcde$ (with $a$, $b$, $c$, $d$, and $e$ *all* being nonzero) such that $abcde = edcba$ and the digits of $abcde$ add to an even multiple of 9.

> ■ The following numbers fit the requirement that $abcde = edcba$ and are either multiples of 18 or 36: 14841, 15651, 16461, 17271, 23832, 24642, 25452, 26262, 32823, 33633, 34443, 35253, 41814, 42624, 43434, 44244, 51615, 52425, 53235, 59895, 61416, 62226, 68886, 69696, 71217, 77877, 78687, 79497, 86868, 87678, 88488, 89298, 95859, 96669, 97479, and 98289.

(d) Find a three digit number $abc$ (with $a \neq 0$ and $c \neq 0$) such that $abc$ is a perfect square and $cba$ is a prime number. (You don't need to show $cba$ is prime for full marks.)

> ■ Any of the following will work: $14 \times 14 = 196 \rightarrow 691$, $19 \times 19 = 361 \rightarrow 163$, and $28 \times 28 = 784 \rightarrow 487$.

(e) Find all numbers of the form $ab$ (with $a \neq 0$ and $b \neq 0$) where $ab$ and $ba$ are *both* prime numbers. (For example, 23 is not a number of this type as 32 is even.)

> ■ There are nine numbers in total that fit the requirements: 11, 13, 17, 31, 37, 71, 73, 79, and 97.

***This question was clearly more difficult than the first two questions, although much easier than the previous year's Question 3. Most students handled the first two parts without difficulty. A few students failed to spot the fact that the number in (c) needed to be even, and gave answers like 12321 (which is an odd multiple of 9). In (d) a common answer was $13 \times 13 = 169$, but 961 is not prime (in fact, in a lovely piece of symmetry it is $31 \times 31$). A similar issue caused grief in part (e), where 19 and 91 were given reasonably often — since $91 = 7 \times 13$ neither number is valid. More common in (e) was the omission of 11.***

## Question 4: 20 marks (All Years)

Suppose for a given number we square each of its digits and sum these squares. Further suppose we repeat this process (noting each sum down in turn) until we either reach a sum of 1 or we reach a sum that we have already noted down.

If we reach a sum of 1, we call our original number a *happy number*. Otherwise our original number is a *sad number*. For example, 13 is a happy number: $1^2 + 3^2 = 10$, and $1^2 + 0^2 = 1$. An example of a sad number is 4 (including 4, our list of sums in this case is 4, 16, 37, 58, 89, 145, 42, 20, and back to 4; as such, all the numbers in this list are sad numbers).

(a) What is the smallest possible happy number?

> ■ 1, since $1^2 = 1$.

(b) Use the fact that 4 is a sad number to show that 11 is also a sad number.

> ■ $1^2 + 1^2 = 2$, and $2^2 = 4$.

(c) Is 19 a happy number or a sad number? Show your working (list the sums you must note down to determine whether 19 is happy or sad).

> ■ $1 + 81 = 82$, $64 + 4 = 68$, $36 + 64 = 100$, and $1 + 0 + 0 = 1$, so 19 is a happy number.

(d) How many 2 digit numbers starting with 1 are happy numbers?

> ■ From the preamble we can see that 10 and 13 are happy numbers and 16 is a sad number. From (a) and (b) we can see that 11 is a sad number and 19 is a happy number. We thus have to consider 12, 14, 15, 17, and 18:
>
> **12:** $1^2 + 2^2 = 5$, $5^2 = 25$, $2^2 + 5^2 = 29$, and $2^2 + 9^2 = 85$ which is the equivalent of 58, which is in the 'chain' for 4. Thus 12 is a sad number.
>
> **14:** $1^2 + 4^2 = 17$ and $1^2 + 7^2 = 50$, which is the equivalent of 5, which is in the 'chain' for 12. Thus 14 is a sad number.
>
> **15:** $1^2 + 5^2 = 26$ and $2^2 + 6^2 = 40$, which is the equivalent of 4, thus 15 is a sad number.
>
> **17:** 17 is in the 'chain' for 14, and as such is a sad number.
>
> **18:** $1^2 + 8^2 = 65$ and $6^2 + 5^2 = 61$ which is the equivalent of 16, which is in the 'chain' for 16. Thus 16 is a sad number.
>
> Thus there are three 2 digit numbers starting with 1: 10, 13, and 19.

A *Harshad number* is a number that is divisible by the sum of its digits. For example, 12 is a Harshad number since $12/(1+2) = 4$. A number is called *twice-Harshad* if the number we get when we divide our original number by the sum of its digits is also Harshad.

(e) Are all single digit numbers greater than zero Harshad? If so, briefly explain why this is, and if not, give an example of a single digit number greater than zero that is not Harshad.

■ Yes, because the sum of a single digit is itself, and when we divide a nonzero number by itself we get 1, a whole number.

(f) Which numbers out of 13, 24, 46, 100, 343, 3100 are not Harshad?

■ $\frac{13}{1+3}$, $\frac{46}{4+6}$, and $\frac{343}{3+4+3}$ are not whole numbers, and thus 13, 46, and 343 are not Harshad.

(g) Which numbers out of 13, 24, 46, 100, 343, 3100 are twice Harshad?

■ $\frac{24}{2+4} = 4$, which is Harshad, so 24 is twice Harshad. $\frac{100}{1} = 100$, so it goes back to itself, so 100 is also twice Harshad. $\frac{3100}{3+1} = 775$, but $\frac{775}{7+5+5}$ is not a whole number, so 3100 is not twice Harshad.

***Like Question 3 most students got a handle on this question overall, and found it a lot easier than the equivalent Question 4 in 2021's competition. The hardest part to get right was (d), where numbers where often mis-identified, typically due to arithmetic errors. The best approach is to identify as soon as possible if the number 'chain' reaches a number in the chain for 4. One can cut down a bit of work by noting that a number of the form $10 \times \mathbf{a} + \mathbf{b}$ (where $\mathbf{a}$ and $\mathbf{b}$ are single digit numbers) is happy or sad if and only if numbers of the form $100 \times \mathbf{a} + 10 \times \mathbf{b} + 0$ and $10 \times \mathbf{b} + \mathbf{a}$ are happy or sad respectively.***

### Question 5: 20 marks (All Years)

(a) Find the non-even prime factor of 272.

■ 17, since $272 = 17 \times 2^4$.

(b) Triangular numbers take the form $1/2n(n+1)$ for some integer $n$ greater than 0. Using this formula, find the 272nd triangular number.

■ $\frac{272 \times 273}{2} = 37128$.

(c) Show that 272 is a *pronic number* — that is, it is the product of two consecutive integers, or twice a triangular number.

■ $16 \times 17 = 272$.

(d) Show that 272 is the sum of four consecutive prime numbers. (Hint: first divide 272 by 4.)

■ $272/4 = 68$. Thus our prime number 'average' must be 68. We know that 63, 65, and 69 are not prime. A quick check of 61, 67, 71, and 73 indicate these numbers are all prime, they are consecutive, and sum to 272, as desired.

(e) Suppose you have constructed an isosceles triangle with a base side of 272 units. Let $h$ be the height of this triangle, $a$ be the area of this triangle, and $s$ be the length of each of this triangle's other two sides. If $h$, $a$, and $s$ are all positive integers, find one set of possible values for $h$, $a$, and $s$. (Hint: remember that the area of a triangle is half its base times its height.)

■ From (b) if we set $h = 273$ we get $a = 37128$. Using Pythagoras we find $s$:

$$\begin{aligned}\left(\frac{272}{2}\right)^2 + 273^2 &= s^2 \\ 136^2 + 74529 &= s^2 \\ 93025 &= s^2 \\ 305 &= s \qquad \text{(ignoring the negative root)}\end{aligned}$$

***This question was a good way to distinguish the top students from the rest. The first four parts are bookwork, and most students handled these quite well. (Note that (c) is trivial if you use the result fount in (a).)***

***It was part (e) where most students struggled. The hint about the area was to prompt students to look to (b), but few students picked this up. Most candidates who did well in part (e) noted that 136 was a multiple of numbers such as 4 or 17, and thus found $\mathbf{h}$ and $\mathbf{s}$ by multiplying a known Pythagorean triple by a scalar factor.***

## Question 6: 20 marks (All Years)

Suppose we have a freight container whose internal dimensions are 2.5 metres wide by 12 metres long by 3 metres high. Into this container may be placed boxes of different sizes.

(a) Consider a metal box with dimensions 200 mm by 200 mm by 200 mm. What is the maximum number of these boxes that we could fit into our freight container?

■ Along the width of the container we can fit 12 boxes ($200 \times 12 = 2400$ but $200 \times 13 = 2600 > 2500$). Similarly, along the length of the container we can fit 60 boxes, while stacking the boxes to the height of the container we can fit 15 boxes. In total we can fit $12 \times 60 \times 15 = 10800$ boxes in the freight container.

(b) Consider another metal box with dimensions 200 mm by 480 mm by 480 mm. If each box has the same orientation inside the container, what is the maximum number of these boxes that we could fit into our freight container? (Hint: the box can have three different orientations.)

■ Our box has one different dimension (200 mm), so consider putting boxes side by side using that dimension. Three options:

(i) Orientate the box so it is 200 mm wide. We then put boxes along the width of the container; from (a) that gives us 12 boxes that way. We then get $12000/480 = 25$ boxes along the length of the container, and we can stack boxes 6 units high in the container since $7 \times 480 = 3360 > 3000$. In total $12 \times 25 \times 6 = 1800$ boxes in the freight container.

(ii) Orientate the box so it is 200 mm long. We then put boxes along the length of the container; from (a) that gives us 60 boxes that way. Once again we can stack boxes 6 units high, and this time we can get 5 boxes along the width (since $6 \times 480 = 2880 > 2500$). In total $60 \times 6 \times 5 = 1800$ boxes in the freight container.

(iii) Orientate the box so it is 200 mm high. We can then stack boxes up to the height of the container; from (a) that gives us 15 boxes that way. Once again we get 25 boxes along the length, and 5 boxes along the width. In total $15 \times 25 \times 5 = 1875$ boxes in the freight container.

So at most we can stack 1875 boxes into the freight container.

(c) Consider a third metal box with dimensions 200 mm by 480 mm by 625 mm. If each box has the same orientation inside the container, what is the maximum number of these boxes that we could fit into our freight container?

■ There are six possible orientations:

| # | W (mm) | L (mm) | H (mm) | W (boxes) | L (boxes) | H (boxes) | Total boxes |
|---|---|---|---|---|---|---|---|
| (1) | 200 | 480 | 625 | 12 | 25 | 4 | 1200 |
| (2) | 200 | 625 | 480 | 12 | 19 | 6 | 1368 |
| (3) | 480 | 200 | 625 | 5 | 60 | 4 | 1200 |
| (4) | 480 | 625 | 200 | 5 | 19 | 15 | 1425 |
| (5) | 625 | 200 | 480 | 4 | 60 | 6 | 1440 |
| (6) | 625 | 480 | 200 | 4 | 25 | 15 | 1500 |

Of these option (6) fits in the most number of boxes.

***This was the first truly difficult question in this year's competition. Most students had a handle on part (a), and made good progress in part (b). Where candidates struggled was in part (c). A table is not necessary, since it can be shown that the boxes as oriented in option (6) fill the container such that there are no gaps in any dimension. However this argument when given by students was not often presented well; the table approach is more methodical and less reliant on a word argument.***

## Question 7: 10 marks (Years 10 and 11 only)

While walking along the beach one day Kitty finds a rather strange treasure map inside a bottle. Each direction on the map is given as a mathematical puzzle.

The directions are as follows:

(a) Starting at the rock shaped like a dragon's head at the north end of the beach, face due east, then rotate 30° anti-clockwise. Walk 5 metres in the direction you are facing.

(b) From the point reached in (a), walk $f(6)$ metres north, then $f(7)$ metres east, then $f(8)$ metres north, and finally $f(9)$ metres east, where $f(n)$ gives you the $n$th Fibonacci number (which is the sum of the previous two Fibonacci numbers; the first 5 numbers in this sequence are $f(1) = 0$, $f(2) = 1$, $f(3) = 1$, $f(4) = 2$, and $f(5) = 3$).

(c) Solve the simultaneous equations $3x + 2y = 22$ and $7x - 3y = 36$, then walk $x$ metres west and $y$ metres south from the point you reached in (b). Start digging!

Being rather clever Kitty finds the treasure in no time at all. Follow the instructions and determine how many metres east and how many metres north the treasure is from Kitty's starting point at the rock. Round your answers to 3 significant figures.

■ We consider each direction in turn:

(a) We construct a standard 30-60-90 triangle, with the hypotenuse having length 5 m. Thus we walk 2.5 m north and approximately 4.33 m east.

(b) Since $f(6) = 5$, $f(7) = 8$, $f(8) = 13$, and $f(9) = 21$, we walk 18 m north and 29 m east.

(c) Multiply the first equation by 3 and the second equation by 2 to get $9x + 6y = 66$ and $14x - 6y = 72$. Add the two equations together and get $23x = 138$. Thus $x = 6$ which gives $y = 2$. Hence we now walk 6 m west and 2 m south.

To three significant places, in total we walk $2.5 + 18 - 2 = 18.5$ m north and $4.3 + 29 - 6 = 27.3$ m east from our starting point to the treasure.

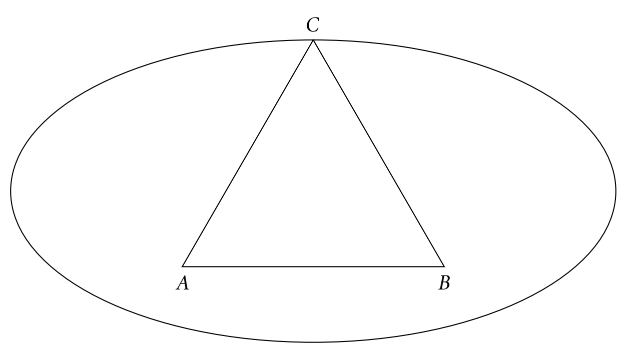
***In general Year 11 students found the going easier here than Year 10 students, but overall the final answer was not often seen. The major issue seemed to be the trigonometry required to find the solution to (a), although mistakes in solving the simultaneous equations given in (c) were not infrequent, and some students also used the wrong Fibonacci numbers in (b).***

## Question 8: 10 marks (Year 11 only)

Note: the formula for the area of the ellipse is $ab\pi$, where $a$ is half the height of the ellipse, and $b$ is half the width of the ellipse.

Suppose we have an ellipse twice as wide as it is tall. An equilateral triangle is drawn inside the ellipse, such that:

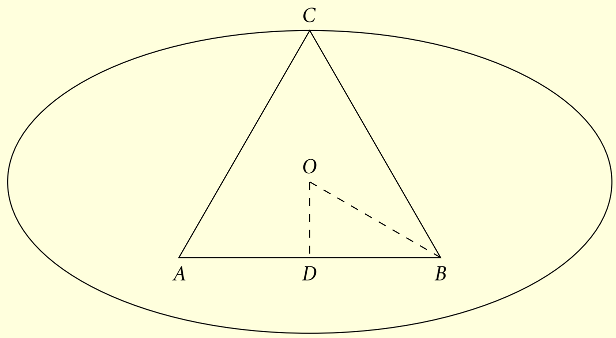
- The centre of the triangle is the centre of the ellipse.
- One of the corners of the triangle ($C$) co-incides with the ellipse's highest point.
- One of the sides ($\overline{AB}$) of the triangle is parallel to the tangent to the ellipse at $C$.



If the area of the ellipse is between 40 and 50 square centimetres and the height of the triangle is a whole number of centimetres, find the area of the ellipse to two decimal places.

- Label the centre of the triangle (and ellipse) $O$.

Suppose the distance from the centre of the triangle to one of its corners is $n$. If we extend a line from the centre of the triangle to the bottommost point of the ellipse, this line will intercept the triangle at point $D$. Since the angle $AOB$ is 120°, the triangle $\triangle OBD$ is a 30-60-90 triangle. Thus the distance from $O$ to $D$ is $n/2$ (since the sine of 30° is 0.5), and so the height of the triangle is $n + n/2$ or $3n/2$.




Using the ellipse area formula, since $n$ is half the height of the ellipse, $2n$ is half the width of the ellipse. Thus the area of the ellipse is $2n^2\pi$.

We then set up a system of inequalities:

$$\begin{aligned} 40 < 2n^2\pi & < 50 \\ \frac{20}{\pi} < n^2 & < \frac{25}{\pi} \\ \sqrt{\frac{20}{\pi}} < n & < \sqrt{\frac{25}{\pi}} \qquad \text{(taking the positive roots only)} \\ \frac{3}{2}\sqrt{\frac{20}{\pi}} < \frac{3n}{2} & < \frac{3}{2}\sqrt{\frac{25}{\pi}} \\ 3.78 < \frac{3n}{2} & < 4.24 \end{aligned}$$

Since $3n/2$ is a whole number, it follows that $3n/2 = 4$. Thus $n = 8/3$.

The area of the ellipse is then $2 \times \frac{8}{3}^2\pi = 44.68\,\mathrm{cm}^2$ (to two decimal places).

***As expected this was the hardest question in the 2022 competition, and very few students made any progress at all. A few students managed to produce the correct answer with little or no working — only partial credit could be given in such cases. Note that in the above solution, if you allow the negative roots in the inequality you get other 'valid' values for n. These values need to be checked to make sure they don't actually work.***