Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

# Junior Mathematics Competition 2023

## Student Report

### Year 9 (Form 3) Prize Winners

**First** Alston Huang, Rangitoto College
**Second** Frank Jin, Scots College
**Third** Ryan Lott, Green Bay High School

**Top 30 (in School Order):**

ChengYu Li, ACG Parnell College
Anas Zara, Avondale College
Ian Chen, King's College
Lachie Turbott, King's College
Anthony Hu, Macleans College
Edgar Xie, Macleans College
Toby Stephen, Otumoetai College
Tiantian Chen, Rangi Ruru Girls' School
Jens (Chun Kiu) Zhao, St Kentigern College
Daniel Li, ACG Parnell College
George Zhao, Botany Downs Secondary College
Kai Li, King's College
Eason Guo, Kristin School
Matthew Kim, Macleans College
Anthony Varun, Naenae College
Yicheng Wang, Pinehurst School
Felix Luo, Rangitoto College
Bruce Zhang, St Paul's Collegiate School
Renish Shah, Avondale College
Miranda Yuan, Burnside High School
Ethan Sheng, King's College
Jack Swainson, Logan Park High School
Daniel Lee, Macleans College
Muhammad Mangera, Newlands Intermediate School
Tamara Nguyen, Queen Margaret College
Alina Chen, St Cuthbert's College
Shawn Zhou, Westlake Boys' High School

### Year 10 (Form 4) Prize Winners

**First** Rudra Patel, Avondale College
**Second** Gwang Ho Kim, Rangitoto College
**Third** Thavanesh Thevar, Avondale College

**Top 30 (in School Order):**

Amy Jing, ACG Parnell College
Xuanwei Chen, Diocesan School for Girls
William Huang, Kristin School
Richie Liu, Kristin School
Janie Kim, Macleans College
Jaymin Corn, Otumoetai College
Jaylen Ling, Rangitoto College
Ariel Liu, St Cuthbert's College
Nish Wood, Tihoi Venture School St Paul
Max McCulloch, Avondale College
Frank Deng, Kristin School
Matthew Lee, Kristin School
Enzo Li, Long Bay College
Wesley Lau, Macleans College
Zefang Fu, Rangitoto College
Aaron Santoso, Sancta Maria College
Celine Yuan, St Cuthbert's College
Huanyao Wang, Westlake Boys' High School
Syed Ahmed Salman, Avondale College
Junyi Guo, Kristin School
Mikey Li, Kristin School
Samuel Chan, Macleans College
Richard Tao, Macleans College
Ruka Kambe, Rangitoto College
Joshua Exon, St Andrew's College
Annabelle Stokes, St Paul's Collegiate School
Samuel Wong, Westlake Boys' High School

### Year 11 (Form 5) Prize Winners

**First** Oscar Prestidge, St Kentigern College
**Second** Daniel Xian, St Kentigern College
**Third** Leo Wang, Rangitoto College

**Top 30 (in School Order):**

Aran Chen, ACG Parnell College
Anlan Gai, ACG Strathallan College
Wesley Teh, Botany Downs Secondary College
Jimmy Cao, Macleans College
Belinda Shi, Macleans College
Alston Yam, Macleans College
Jay Zhao, Macleans College
Phineas Ou, Mount Roskill Grammar School
Yujin Sung, St Cuthbert's College
Chenghao Li, ACG Parnell College
Rohan Kumar, Avondale College
Aaron Kwak, Christ's College
Sunny Liu, Macleans College
Chenyou Song, Macleans College
Tony Yu, Macleans College
Alexander Lau, Middleton Grange School
Jiahong Yu, Pinehurst School
Jasper Carran, St Peter's College (Epsom)
Allen Weng, ACG Parnell College
An-Jhuo Lee, Bayfield High School
Ryan Fan, Kristin School
Kenneth Luo, Macleans College
Haowen Xie, Macleans College
Bruce Zhang, Macleans College
Ava Poynter, Mount Albert Grammar School
Chloe Seo, Rangitoto College
Maxwell Clarke, Wellington High School

As always only one method is shown. Often several methods exist. We don't guarantee that any method shown is the best or fastest in any case.

## Question 1

Farah has an appointment across town from her office. Normally she could drive to it by just going North a certain amount then West a certain amount, but today the town is full of traffic works and many of the roads are blocked off.

Fortunately her car's GPS can navigate her through the blocked road system. From her office, it tells her to take the route described in the box to the right.

*(1) From your office, drive 51 metres North.*

*(2) Drive 75 metres East.*

*(3) Drive 59 metres North.*

*(4) Drive 120 metres West.*

*(5) Drive 45 metres South.*

*(6) Drive 111 metres West to your destination.*

(a) How many metres does Farah drive in her car in total?

■ $51 + 75 + 59 + 120 + 45 + 111 = 461$ metres.

(b) If none of the roads were blocked and Farah could take her normal route (so North a certain amount then West a certain amount), how far in metres would she have had to have driven to get to her appointment?

■ $(51 + 59) - 45 = 65$ metres North; $(120 + 111) - 75 = 156$ metres West. So $65 + 156 = 221$ metres in total.

(c) Suppose there was a road that went Northwest from Farah's office to where her appointment was. How far in metres would she have had to have driven between her office and where her appointment was if she took this road?

■ We can create a right-angled triangle; with this we can use the theorem of Pythagoras. So $\sqrt{65^2 + 156^2} = 169$ metres.

***In general Year 9 students found most of this question straightforward. Some students failed to total the two subtotals in (b). While a correct answer to (c) was not common, we expected this to be the case.***

## Question 2

Using the constraints listed below, find values for the variables in the following 3 by 3 table:

| | | |
|---|---|---|
| $a$ | 3 | $c$ |
| 4 | $e$ | $f$ |
| $g$ | $h$ | $i$ |

(a) Each of 1, 2, 3, 4, 5, 6, 7, 8, and 9 appear exactly once in the table.

(b) $e$ is even while $h$ is odd.

(c) $a < c$ and $g > i$.

(d) There is no row in the table with an odd number of even numbers.

(e) The sum of the first row is 9, while the sum of the last row is 19.

■
- Since the sum of the first row is 9 we can see that $a + c = 6$. Since 4 has already been used and $a < c$ this means $a = 1$ and $c = 5$.
- Since $6 + 7 + 8 > 19$ we must have 2 in the last row. This also means the other numbers in the last row are 8 and 9.
- Since $h$ is odd we now have $h = 9$. Since $g > i$ we now have $g = 8$ and $i = 2$.
- Only 6 and 7 are left; since $e$ is even this means $e = 6$ and $f = 7$.

In summary $a = 1$, $c = 5$, $e = 6$, $f = 7$, $g = 8$, $h = 9$, and $i = 2$. To the right the answer is presented in table form.

| | | |
|---|---|---|
| 1 | 3 | 5 |
| 4 | 6 | 7 |
| 8 | 9 | 2 |

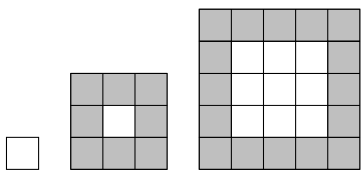
***This was the best answered question in 2023 by both Year 9 and Year 10 students. The only mathematical knowledge needed to answer the question is arithmetic, ordering, and the concept of odd and even numbers. The most common mistakes were misreading constraints (b) and (c) — for example, some students stated in their answers that e was odd or that c < a. For the most part this meant these students had everything correct except for two numbers being switched around. Note that you don't need to use constraint (d) to find the above solution.***

## Question 3

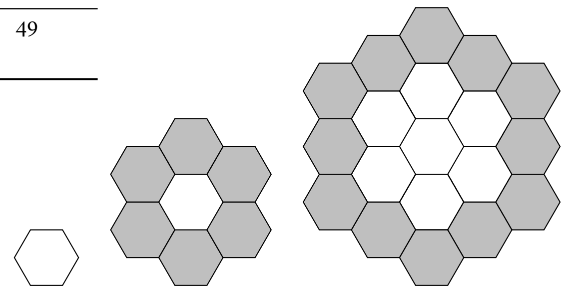
Hahona builds up a polygon by taking a square and surrounding it by other squares of the same width, height, and orientation such that each corner or edge of the original square is adjacent to a corner or edge of another square. He then repeats the process, at each step creating a larger polygon. Hahona also does the same thing with regular hexagons — see the figures to the right.

In the below table $n$ represents a stage of the polygon construction (with $n = 1$ representing the base square or regular hexagon), $s$ represents the number of squares used in this stage, and $h$ represents the number of hexagons used in this stage.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $s$ | 1 | 9 | | 49 | |
| $h$ | 1 | 7 | 19 | | |



$n = 1$ $n = 2$ $n = 3$



$n = 1$ $n = 2$ $n = 3$

(a) Find $s$ for

(i) $n = 3$

■ $n = 3$ gives $s = 25$.

(ii) $n = 5$

■ $n = 5$ gives $s = 81$. (In general $s$ represents the $n$th odd square number.)

(b) Find $h$ for

(i) $n = 4$

■ For each successive $n$ we add $6(n-1)$ hexagons. Since for $n = 3$ we have 19 hexagons, for $n = 4$ we have $6(n-1) = 18$, so $h = 19 + 18 = 37$.

(ii) $n = 5$

■ From (i) $n = 5$ gives $6(n-1) = 24$ and $h = 37 + 24 = 61$.

(c) Find the smallest $n$ such that

(i) $s > 300$

■ If $n = 9$ then $s = 289$; if $n = 10$ then $s = 361$. So we want $n = 10$.

(ii) $h > 200$

■ If $n = 8$ then $h = 169$; if $n = 9$ then $h = 217$. So we want $n = 9$.

(d) If the perimeter of the regular hexagon is 6 cm, find the perimeter of the polygon constructed using the regular hexagon when $n = 5$.

■ For $n = 1$ the perimeter is 6 cm. For $n = 2$ the perimeter is 18 cm, while for $n = 3$ the perimeter is 30 cm. Thereafter for each 'side' of the constructed polygon we add 1 hexagon that only has 2 exterior faces, so each 'side' adds 2 cm to the perimeter. Hence every time we increase $n$ by 1 we add $6 \times 2 = 12$ cm to our perimeter. Thus for $n = 4$ the perimeter is 42 cm, and for $n = 5$ the perimeter is 54 cm.

***This was the first question all years answered, and for the most part the underlying mathematics was understood by most students and overall most students gained some marks here. A lot of students opted for an algebraic approach throughout the question, although there was plenty of evidence for students using diagrams to find their answers (and a few students answered (d) using this approach). In part (d) the final answer was seen reasonably often; however a thorough explanation for this answer was often lacking, so full marks for this question was not that common.***

## Question 4

In this question $n$ is an integer greater than 0, and $p$ is a prime number.

For a given $n$ let $p$ be the $n$th prime number and $s(n)$ be the distance from $p$ to the square number closest to $p$. For example, if $n = 4$ then $p = 7$. The closest square numbers to 7 are 4 and 9, and since 9 is closer to 7 than 4 we have $s(4) = 9 - 7 = 2$.

(a) For the following values of $n$ and $p$ find $v$ (the largest square number smaller than $p$), $w$ (the smallest square number larger than $p$), and $s(n)$:

(i) $n = 7$, $p = 17$

■ $n = 7$ and $p = 17$ gives $v = 16$, $w = 25$, and $s(n) = 1$ (since $p$ is closer to $v$ than $w$).

(ii) $n = 10$, $p = 29$

■ $n = 10$ and $p = 29$ gives $v = 25$, $w = 36$, and $s(n) = 4$.

(b) Find the smallest value of $n$ such that $s(n) > 5$.

■ If $n = 14$ we have $p = 43$, where $v = 36$ and $w = 49$, thus $s(n) = 6$.

A function similar to $s(n)$ is $c(n)$, which for a given $n$ is the distance from $p$ to the cube number closest to $p$ (where again $p$ is the $n$th prime number). For example, the closest cube numbers to $p = 7$ are 1 and 8, and since 8 is closer than 1 $c(4) = 8 - 7 = 1$.

(c) For the following values of $n$ and $p$ find $y$ (the largest cube number smaller than $p$), $z$ (the smallest cube number larger than $p$), and $c(n)$:

(i) $n = 7$, $p = 17$

■ $n = 7$ and $p = 17$ gives $y = 8$, $z = 27$, and $c(n) = 9$.

(ii) $n = 16$, $p = 53$

■ $n = 16$ and $p = 53$ gives $y = 27$, $z = 64$, and $c(n) = 11$.

(d) Find the smallest value of $n$ such that $c(n) > 10$.

■ If $n = 13$ we have $p = 41$, where $y = 27$, $z = 64$ once again, and $c(n) = 14$.

(e) For $n = 1$, $p = 2$ and $s(1) = c(1)$. Can you find another value for $n$ such that $s(n) = c(n)$? If so, list $n$, $p$ (the $n$th prime number), $s(n)$, and $c(n)$. If not, briefly explain why no such $n$ exists.

■ 64 is a both a square number and a cube number. Thus for $17 \leq n \leq 20$ we have the following:

- $n = 17 \rightarrow p = 59$, and $s(n) = c(n) = 5$. (Clearly the closest cube / square is 64.)
- $n = 18 \rightarrow p = 61$, and $s(n) = c(n) = 3$. (See above.)
- $n = 19 \rightarrow p = 67$, and $s(n) = c(n) = 3$. (See above.)
- $n = 20 \rightarrow p = 71$, and $s(n) = c(n) = 7$. (See above.)

In general there are an infinite number of integers that are both square numbers and cube numbers: for a given positive integer $x$ we have $x^6$ which is both a square number $(x^3)^2$ and a cube number $(x^2)^3$.

This allows us to find some further examples:

- If $x = 3$ then $x^6 = 729$, and when $127 \leq n \leq 133$ we have $s(n) = c(n)$.
- If $x = 4$ then $x^6 = 4096$, and (for instance) when $n = 650$ then $p = 4057$ and $s(n) = c(n) = 39$.

There are further possibilities than the above.

***This was a more technical question than Question 3, and this was reflected in the slight decrease in mean grade from the latter. Note that mathematically speaking distance is always non-negative. Part (e) was poorly answered overall — only a minority of students realised a number could be both a square and a cube, but most of those who did proceeded to give a correct answer. The counting of primes also tripped up quite a few students, with many missing out on a prime. This was most commonly seen in (b), where a number of students listed 43 as the thirteenth prime and not the fourteenth.***

## Question 5

For a set of **positive real** numbers $x_1, x_2, \ldots, x_n$ we define three functions $A$, $G$, and $H$:

$$A(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n} \qquad H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

The function $A$ refers to the *arithmetic* mean of $x_1, x_2, \ldots, x_n$, while $G$ and $H$ refer to the *geometric* and *harmonic* means of our set respectively. For example, for the set $\{1, 2, 3, 4\}$, $n = 4$, and we have $A(1, 2, 3, 4) = 2.5$, $G(1, 2, 3, 4) = \sqrt[4]{24} = 2.21$ (to two decimal places), and $H(1, 2, 3, 4) = 1.92$. (When we take the $n$th root for $G$ we ignore the negative root if $n$ is even.)

(a) Find $A(4, 6.3, 7.2)$, $G(4, 6.3, 7.2)$, and $H(4, 6.3, 7.2)$, rounding to two decimal places if necessary. (Here $n = 3$.)

■ To two decimal places, $A(4, 6.3, 7.2) = 5.83$, $G(4, 6.3, 7.2) = 5.66$, and $H(4, 6.3, 7.2) = 5.48$.

For the set $\{2, 4, 6, 8\}$, $A(2, 4, 6, 8) = 5$, $G(2, 4, 6, 8) = 4.42$ (to two decimal places), and $H(2, 4, 6, 8) = 3.84$. These are twice the values of $A(1, 2, 3, 4)$, $G(1, 2, 3, 4)$, and $H(1, 2, 3, 4)$ respectively.

(b) For any **positive real** number $b$, show that:

(i) $A(bx_1, bx_2, \ldots, bx_n) = bA(x_1, x_2, \ldots, x_n)$

■

$$\begin{aligned} A(bx_1, bx_2, \ldots, bx_n) &= \frac{1}{n}(bx_1 + bx_2 + \cdots + bx_n) \\ &= \frac{1}{n}b(x_1 + x_2 + \cdots + x_n) \\ &= b\left(\frac{1}{n}(x_1 + x_2 + \cdots + x_n)\right) \\ &= bA(x_1, x_2, \ldots, x_n) \end{aligned}$$

(ii) $G(bx_1, bx_2, \ldots, bx_n) = bG(x_1, x_2, \ldots, x_n)$

■

$$\begin{aligned} G(bx_1, bx_2, \ldots, bx_n) &= (bx_1 \times bx_2 \times \cdots \times bx_n)^{1/n} \\ &= (b^n \times x_1 \times x_2 \times \cdots \times x_n)^{1/n} \\ &= (b^n)^{1/n}(x_1 \times x_2 \times \cdots \times x_n)^{1/n} \\ &= b(x_1 \times x_2 \times \cdots \times x_n)^{1/n} \\ &= bG(x_1, x_2, \ldots, x_n) \end{aligned}$$

(iii) $H(bx_1, bx_2, \ldots, bx_n) = bH(x_1, x_2, \ldots, x_n)$

■

$$\begin{aligned} H(bx_1, bx_2, \ldots, bx_n) &= \frac{n}{\frac{1}{bx_1} + \frac{1}{bx_2} + \cdots + \frac{1}{bx_n}} \\ &= \frac{n}{\frac{1}{b}(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n})} \\ &= \frac{bn}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \\ &= b\left(\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}\right) \\ &= bH(x_1, x_2, \ldots, x_n) \end{aligned}$$

(c) Is it possible to find a set of 10 numbers such that their arithmetic, geometric, and harmonic means are all equal? If so, list such a set. If not, briefly explain why no such set exists.

■ Yes. Let be $S$ the set of our 10 numbers. If the members of $S$ are all the number 10, then $A(S) = (10 \times 10)/10 = 10$, $G(S) = (10^{10})^{1/10} = 10$, and $H(S) = 10/(10 \times 1/10) = 10/1 = 10$.

(Note that the key thing here is for all the members of $S$ to be the same number.)

***This very technical, book work question proved too much for the vast majority of students, and was the worst answered question for the most part. Many students could answer (a) without too much difficulty, but the algebraic proofs in (b) were beyond the ability of many. Part (c) was similarly poorly answered although the actual answer is relatively simple.***

## Question 6

In this question, we only get out of a lift if we are going into another lift or we have reached our destination.

JMC Towers operates two lifts. Each lift follows a set of rules as to which floors it will stop at:

*(1)* When going up, lift A can only stop at every fourth floor from where it started, but when going down it can only stop at every fifth floor.

*(2)* When going up, lift B can only stop at every seventh floor from where it started, but when going down it can only stop at every ninth floor.

Suppose that JMC Towers has 30 storeys, with the ground floor being numbered 0 and the highest floor being numbered 29, and assume that neither lift provides access to the basement.

(a) If you get into lift A on the ground floor, what is the highest floor you can reach if you only use this lift (going up only)?

■ $4 \times 7 = 28$, so you can only get to this floor, since $4 \times 8 = 32 > 29$.

(b) If you get into lift B on the 29th floor, what is the lowest floor you can reach if you only use this lift (going down only)?

■ $9 \times 3 = 27$, so you can only get to the $29 - 27 = 2$nd floor, since $9 \times 4 = 36 > 29$.

If you wanted to go from the ground floor to the 11th floor without ever going down, one way to do this (which we call a *path*) is to get into lift A going up, get out at the 4th floor when it stops for the first time, then get into lift B going up, and get out when it stops the first time. A shorthand way of stating this is AU1, BU1. To get to the 15th floor from the ground floor (only going up), there are now two distinct paths: AU2, BU1 or AU1, BU1, AU1; we say the first path has two stops, and the second three stops. If we went down from the 14th floor to the ground floor, one path would be AD1, BD1.

(c) Find a path to get to the 29th floor from the ground floor if you can *optionally* change lifts every time a lift reaches a floor, and can only go up.

■ We have the Diophantine equation $4a + 7b = 29$, which when $a \geq 0$ and $b \geq 0$ has the only integer solution of $a = 2, b = 3$. Any of the following paths will therefore work:

AU2, BU3
AU1, BU3, AU1
AU1, BU2, AU1, BU1
AU1, BU1, AU1, BU2

BU3, AU2
BU2, AU2, BU1
BU2, AU1, BU1, AU1
BU1, AU2, BU2
BU1, AU1, BU2, AU1
BU1, AU1, BU1, AU1, BU1

(d) Is the path you found in (c) the only path you could take with the same conditions applying? If so, briefly explain why this is the case. Otherwise, list another path with the same conditions as in (c).

■ From (c) we can see we have multiple paths, so anything from that list that wasn't chosen in (c) will suffice here.

(e) There exists a path to the ground floor from the 29th floor that takes seven stops, providing that you *must* change lifts every time a lift reaches a floor it can stop at, and you can either go up or down every time you change lifts. Find such a path.

■ AD1, BD1, AU1, BD1, AU1, BD1, AD1.

***Compared to Question 5 this was answered reasonably well. The question setters' fears that the definitions given in the question would confuse most students were ill-founded; most students used the shorthand notation for travelling in a lift without any issue. Some students struggled with 'out by one' errors, and the wording in (c), (d), and (e) tripped up some students who began their travels in an impossible direction. In general though the only hard part of the question for those who attempted it was (e), where the lack of options meant that many students (sensibly) gave up before finding the correct answer. (We believe the printed answer to be the only correct one.)***

## Question 7

Sierpiński's triangle is a variation of Pascal's triangle: we take each number in the latter structure, and if said number is odd, the corresponding number in the former structure is 1; otherwise it is 0. As such, since the first 5 lines of Pascal's triangle are {1}, {1, 1}, {1, 2, 1}, {1, 3, 3, 1}, and {1, 4, 6, 4, 1}, the first 5 lines of Sierpiński's triangle are {1}, {1, 1}, {1, 0, 1}, {1, 1, 1, 1}, and {1, 0, 0, 0, 1}, respectively.

(a) List the numbers in the 6th line of Sierpiński's triangle.

■ 1 1 0 0 1 1

If we take a line of numbers in Sierpiński's triangle, we can use each number as a digit in a binary number, which can then be converted to a decimal number. For example, the 5th line in Sierpiński's triangle reads 1, 0, 0, 0, and 1, which can be represented as the binary number 10001, which in decimal is 17 (since $17 = 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$).

(b) Find the binary representation of the 7th line of Sierpiński's triangle, then convert it into a decimal number. (List both your binary and decimal numbers.)

■ The seventh line is 1 0 1 0 1 0 1, which is $2^6 + 2^4 + 2^2 + 2^0 = 85$.

(c) Briefly explain why the decimal representation of every line of Sierpiński's triangle is odd.

■ The last number in each line of Pascal's triangle is always 1, so this is also true of the same line in Sierpiński's triangle. Hence the sum that creates our decimal representation always ends in $2^0 = 1$ which is odd. Since all the other parts of the sum are even ($2^x$ is even for integer $x > 0$) the sum of the binary expansion must therefore be odd.

(d) The decimal representation of the 5th line is a prime number. What is the next line of Sierpiński's triangle with a prime decimal representation? (List the line number, the binary representation, and its decimal form.)

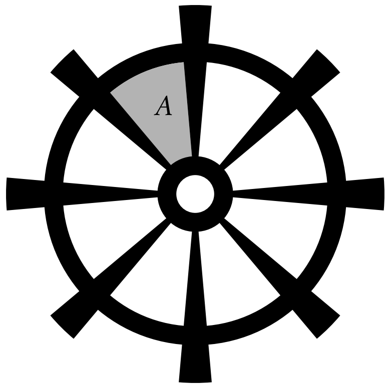
■ From the text the decimal representation of the fifth line is 17. The sixth line sums to 51 ($= 17 \times 3$), the seventh 85 (a multiple of 5), and the eighth 255 (also a multiple of 5). The ninth line of Pascal's triangle is 1 7 21 35 35 21 7 1. The equivalent line in Sierpiński's triangle is 1 0 0 0 0 0 0 0 1 which equates to 257, a prime number.

***This question was a follow up to a question in part one about Pascal's triangle. (Actually it was written first, and the question in part one was added so we knew top candidates had some knowledge of it.) In an earlier competition a question about binary numbers was given much earlier on, and was poorly answered overall; placing such a question later in the competition seemed to have the desired effect, and we were quite pleased to see this question handled reasonably well. In part (c) almost every student who answered the question picked up on the outer edges of both triangles being 1. Note that every line in each triangle is symmetric, so it doesn't matter if you set the first number in Sierpiński's triangle to be $2^0$ or the last.***

## Question 8

For a school art project Kurt is designing a poster with a nautical theme. On the poster he will draw an abstract representation of a sailing ship's wheel, which can be seen to the right.

The wheel consists of two rings: an inner ring with an inner radius of 2 cm and an outer radius of 4 cm, and an outer ring with an inner radius of 14 cm and an outer radius of 16 cm. There are also 8 spokes, each of which is constructed using a 10° sector of a circle with radius 20 cm, placed so that the sharp end of the spoke is at the middle of the wheel, and then cut off at the outer radius of the inner ring.




(a) To 2 decimal places find the area of the outer ring, including the areas that intersect with the spokes.

■ $16^2\pi - 14^2\pi = 188.49\,\text{cm}^2$ to 2 decimal places.

(b) To 2 decimal places find the area of one of the wheel's spokes, including the area that intersects with the outer ring.

■ 10 degrees is $\frac{1}{36}$ of $2\pi$. $\frac{20^2\pi}{36} = 34.91\,\text{cm}^2$ to 2 decimal places. The bit inside the circle is $\frac{2^2\pi}{36}$, so in total $\frac{20^2\pi}{36} - \frac{2^2\pi}{36} = 34.58\,\text{cm}^2$ to two decimal places.

(c) To 2 decimal places find the area of the empty section of wheel labelled $A$ and shaded grey on the diagram.

■ A line from the centre of the wheel to the middle of the shaded segment furthest from the centre is 14 cm long. Since there are 8 spokes, the degrees of 'sweep' the area takes up is $(360 - 10 \times 8)/8 = 35$. If we ignore the inner segment it then has area $(35/360) \times 14^2\pi = 59.86\,\text{cm}^2$ to 2 decimal places. Taking away the inner area of $(35/360) \times 4^2\pi$ we get $54.98\,\text{cm}^2$ to two decimal places.

(d) To 2 decimal places find the area of the ship's wheel (the total area of every part of the diagram shaded black).

■ Since we have the area of $A$, if we subtract 8 lots of that from a circle of radius 16 cm and also subtract the inner circle with radius 2 cm we need only add the 'outer' parts of the 8 spokes. The outer part of each spoke is $\frac{20^2\pi}{36} - \frac{16^2\pi}{36} = 12.57\,\text{cm}^2$ to 2 decimal places. In total $16^2\pi + 8(\frac{20^2\pi}{36} - \frac{16^2\pi}{36}) - 8(\frac{35}{360}14^2\pi - \frac{35}{360}4^2\pi) - 2^2\pi = 452.39\,\text{cm}^2$ to two decimal places.

***For Year 11 students this was the hardest question in this year's competition (although seemingly slightly easier than Question 8 last year), and no student received full marks for this question. Most students who attempted the question could answer the first part correctly, but with each succeeding part of the question fewer and fewer students managed correct answers. This was likely both due to difficulty (adding and subtracting complex geometric shapes can get out of hand quite quickly) and time issues (this year's competition was quite long).***