The University of Otago

Junior Mathematics Competition

2024 Student Report

Department of Mathematics and Statistics — Te Tari Pākarau me te Tatauraka

University of Otago
Ōtākou Whakaihu Waka

Year 9 Prize Winners

First Andrew Law, Westlake Boys' High School
Second Vani Singh, Epsom Girls' Grammar School
Third Frank Jin, Scots College

Top 30 (in School Order):

Timothy Yang, ACG Strathallan College
Nicole Yu, Botany Downs Secondary College
Logan Gray, King's High School
Luo Yuan Bao, Macleans College
Charles Jiang, One Tree Hill College
Gordon Peng, Rangitoto College
Liam Javier, Rosmini College
Ruishan (Bella) Chen, St Kentigern College
Lisa Murata Gutierrez, Takapuna Grammar School
Logan Murphie, Belmont Intermediate School
Johnny Huang, Burnside High School
Conor Kerr, King's High School
Roy Luo, Macleans College
Xuan Li, Rangitoto College
Stephanie Yang, Rangitoto College
Zilong Chen, St Andrew's College
Chloe Huang, St Kentigern College
Chengyuan Li, Westlake Boys' High School
Mevuni Ariyasinghe, Botany Downs Secondary College
Xiangyun Tan, John Paul College
Sophie Xu, Kristin School
Eric Shen, Macleans College
Dingding Mao, Rangitoto College
Dylan Yin, Rangitoto College
Bella Zeng, St Cuthbert's College
Dylan Wang, St Peter's School (Cambridge)
Max Tai, Whitby Collegiate

Year 10 Prize Winners

First Leon Wu, Burnside High School
Second Ryan Lott, Green Bay High School
Third Bella McLauchlan-Hillary, Waikato Diocesan School for Girls

Top 30 (in School Order):

George Zhao, Botany Downs Secondary College
Juliette Colquhoun, Green Bay High School
Eason Guo, Kristin School
Daniel Lee, Macleans College
Edgar Xie, Macleans College
Cesar Xu, Pinehurst School
Felix Luo, Rangitoto College
Chen Sun, St Kentigern College
William Fan, Westlake Boys' High School
Finley Macfarlane, Cashmere High School
Daniel Zhang, Kaili Home School
Isabelle Ning, Kristin School
Eric Liu, Macleans College
Jiaxin Linda Chen, Pinehurst School
Tiantian Chen, Rangi Ruru Girls' School
Cassandra Thompson, St Cuthbert's College
John An, Takapuna Grammar School
Anthony Meng, Westlake Boys' High School
Matilda Aprea, Epsom Girls' Grammar School
Ethan Sheng, King's College
Benjamin Chan, Macleans College
Isla Wang, Macleans College
Yicheng Wang, Pinehurst School
Alston Huang, Rangitoto College
Jackie Xu, St Cuthbert's College
Shia Abdul-Coley, Tauranga Boys' College
Xiaoyang Shawn Zhou, Westlake Boys' High School

Year 11 Prize Winners

First Gwang Ho Kim, Rangitoto College
Second Junyi Guo, Kristin School
Third Richard Tao, Macleans College

Top 30 (in School Order):

Xuanwei Chen, ACG Parnell College
Thavanesh Thevar, Avondale College
Frank Deng, Kristin School
Richie Liu, Kristin School
Wesley Lau, Macleans College
Cindy Cao, Pinehurst School
Kyle Tac, Rangitoto College
Luke Manson, St Andrew's College
Karol Zhang, St Kentigern College
Victor Tai, ACG Parnell College
Hamish Patel-Smith, King's College
Jason He, Kristin School
Emily Chan, Macleans College
Lawrence Wen, Macleans College
Jack Chen, Pinehurst School
Yanchen Tan, Rangitoto College
Zara Toes, St Cuthbert's College
Hamish Bell, St Peter's College (Epsom)
Rudra Patel, Avondale College
Aidan Blakie, King's High School
Matthew Lee, Kristin School
Gisele Chong, Macleans College
Yixiang Xu, Macleans College
Alexander Bai, Rangitoto College
Alex Manson, St Andrew's College
Celine Yuan, St Cuthbert's College
Alexander Lee, Westlake Boys' High School

As always only one method is shown. Often several methods exist. We don't guarantee that any method shown is the best or fastest in any case.

Question 1: 10 marks (Years 9 and below only)

Joanna has received a $250 000 inheritance from her rich grandmother. She plans to invest some of it, and she will spend the other 12.5% of the money.

(a) Write down 12.5% as a decimal.

■ 0.125.

(b) Using the number you wrote down in (a), in one line write down the working needed to show that the amount she will spend is $31 250.

■ $0.125 \times \$250000 = \$31250$.

(c) Joanna decides to use 3/20 of the spending money on a holiday. How much will she spend on the holiday?

■ $3/20 \times \$31250 = \$4687.50$.

(d) For her holiday, Joanna allocates $750 for travel expenses. If she spends 14 nights of her on holiday at an hotel costing $150 a night, how much of her holiday money will she have after accounting for her travel and accommodation costs?

■ $\$4687.50 - \$750 - \$150 \times 14 = \$1837.50$.

(e) What fraction of the original $250 000 Joanna inherited is she spending on the holiday? Give your answer in its simplest fraction form.

■ $\$4687.50/\$250000 = 3/160$.

This was mostly well done by most Year 9 students, with a higher average grade for this question than for the equivalent question last year. Too many students rounded their answers to (c) and (d) to one decimal place instead of giving their answers to the two decimal places customary for monetary amounts. A common mistake in (e) was to use the total found in (d) rather than in (c).

Question 2: 10 marks (Years 10 and below only)

First **copy** the square array shown below into your answer book, then find nine consecutive natural numbers and fill them into the square array in your answer book, such that the sum of the three numbers in each row, column, and diagonal of the array is equal to 60.

■

17	22	21
24	20	16
19	18	23

This variant of the classic 'magic square' question was done reasonably well by a large number of Year 9 and Year 10 students. Clearly there were many students who had previously encountered a magic square, and it is likely that said students had little trouble compared to the students who had never seen this construction before.

In a few cases students gave what is called a 'semi-magic square,' where at least one of the diagonals doesn't add to 60, even though all the rows and columns do. In those cases students were given partial credit.

Question 3: 20 marks (All Years)

2024 can be written as $2^3 \times 11 \times 23$. Put another way, 2024 is of the form p^3qr, where p, q, and r are distinct prime numbers and $r = 2q + 1$. (This definition of p, q, and r is used throughout (a), (b), (c), and (d).)

(a) Find the smallest number of the form p^3qr.

■ $2^3 \times 3 \times 7 = 168$.

(b) If $p = 2$ find the smallest number of the form p^3qr larger than 2024.

■ $2^3 \times 23 \times 47 = 8648$.

(c) Is the number you found in (b) the smallest number of the form p^3qr larger than 2024 (allowing for $p \neq 2$)? If it is, briefly explain why there is no number of the form p^3qr larger than 2024 but smaller than your answer for (b). If it isn't, give another number of the form p^3qr larger than 2024 but smaller than the number you found in (b).

■ No, there is another number of the form p^3qr larger than 2024 but smaller than 8648. For example, $5^3 \times 3 \times 7 = 2625$, $7^3 \times 2 \times 5 = 3430$, and $3^3 \times 11 \times 23 = 6831$ are all smaller than 8648. [For full credit in this part only one example is needed.]

(d) Find all numbers of the form p^3qr smaller than 2024. You do not need to include the number you listed in (a).

■ Apart from 168 we have $3^3 \times 2 \times 5 = 270$, $2^3 \times 5 \times 11 = 440$, and $3^3 \times 5 \times 11 = 1485$.

2024 is the 39th time the Junior Mathematics Competition has run. 39 is the sum of 5 consecutive prime numbers:

$$3 + 5 + 7 + 11 + 13 = 39.$$

(e) Find all other numbers smaller than 100 that are the sum of 5 consecutive prime numbers.

■ Apart from 39 we have $2 + 3 + 5 + 7 + 11 = 28$, $5 + 7 + 11 + 13 + 17 = 53$, $7 + 11 + 13 + 17 + 19 = 67$, and $11 + 13 + 17 + 19 + 23 = 83$.

(f) Which of the numbers you found in (e) are also prime numbers?

■ Of the numbers found in (e) 53, 67, and 83 are also prime. (28 is $2^2 \times 7$.)

This question contains two related sections. The first section (comprising parts (a) through (d)) was on the whole more poorly done than the last section (comprising (e) and (f)). We suspect a lack of careful reading of the question was the cause of this. Overall however most students found this question slightly easier than the equivalent question in last year's competition.

In the first four parts, it was reasonably common to see students giving values of p, q, or r that were not prime numbers, and it was not uncommon to see p = q or similar equalities occur. In a few cases some students set one of p, q, or r to 1, which by the definition of a prime number on the front page was not allowed. (The question is considerably less interesting if the set of prime numbers includes 1.)

Question 4: 20 marks (All Years)

Books published before 2007 have a string of 10 digits on them called an *International Standard Book Number* or *ISBN*, that identifies the book. Each of the first 9 digits in the string is a number between 0 and 9 inclusive, and the 10th digit is either a number between 0 and 9 inclusive or an X.

In order for such a string to be an ISBN, it must satisfy an additional condition. Let x_1 denote the first digit, x_2 the second digit, etc. The last digit x_{10} must be the remainder on dividing

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 8x_8 + 9x_9$$

by 11. (If the remainder is 10, then x_{10} is X.)

(a) Briefly explaining your answer, is the string 038696617X an ISBN?

> ■ No, because $0 + 2 \times 3 + 3 \times 8 + 4 \times 6 + 5 \times 9 + 6 \times 6 + 7 \times 6 + 8 \times 1 + 9 \times 7 = 248$ which has remainder 6 when divided by 11.

(b) The 8th digit of the ISBN printed on a book is illegible. The ISBN reads 0321149Z20, where the 'Z' is unknown. Find Z.

> ■ Here $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 9x_9 = 126$. Since $x_{10} = 0$, we need $126 + 8x_8$ to be a multiple of 11. Brute force gives $x_8 = 9$.

(c) A common error made when inputting numerical data is a "transposition" error, where two adjacent digits get swapped, e.g. someone trying to type "68" might accidentally type "86" instead. Show that if the 4th and 5th digits (which are different) are transposed when typing out an ISBN, the resulting string of digits is not an ISBN.

> ■ Suppose $y = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 8x_8 + 9x_9 + x_{10}$ is a valid ISBN (where we let $x_{10} = 10$ if its value is normally X). It is thus divisible by 11: $y = 11a$ for some a. Let $z = x_1 + 2x_2 + 3x_3 + 4x_5 + 5x_4 + 6x_6 + 7x_7 + 8x_8 + 9x_9 + x_{10}$. If z is also a valid ISBN then there exists b such that $z = 11b$.
>
> Since $z - y = 11b - 11a = 11(b - a)$, it follows that $z - y$ is a multiple of 11. But $z - y = x_4 - x_5$, which cannot be divisible by 11 since $x_4 \neq x_5$ and $-9 \leq x_4 - x_5 \leq 9$. It follows that z cannot be a valid ISBN after all.

This was the first 'hard' question of this year's competition, with some level of algebraic knowledge needed to complete the question. Nonetheless it was not uncommon to see students successfully answer the first two parts of the question. The bulk of the difficulty here is in part (c), where for the most part students struggled. Quite a few students were able to point out that when switching the digits the difference between the two ISBN numbers is a multiple of 11, but then failed to explain why this was not possible. For full marks, an explanation needs to be given as to why a difference of a multiple of 11 is not possible when $\mathbf{x}_4$ and $\mathbf{x}_5$ are both less than 11.

Question 5: 20 marks (All Years)

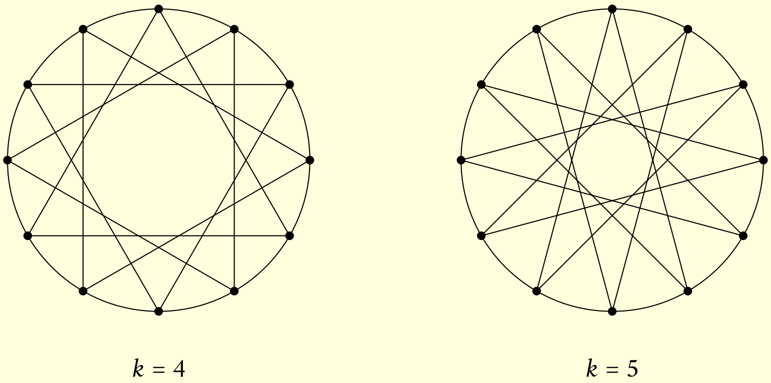
12 points are marked around a circle. A k-star is drawn as follows:

> *(1)* Choose any point.
>
> *(2)* From your current point, draw a straight line from that point to the kth point coming after (clockwise).
>
> *(3)* If the new point doesn't have a line already coming from it, repeat step (2), otherwise move to the next step.
>
> *(4)* If all points are incident to two lines, the k-star is complete. Otherwise choose a point without any lines going to it and repeat the process from step (2).

Note that when drawing the k-star we never draw a line between any two points more than once.

(a) Draw the 4-star and the 5-star.

■



(b) Can the same star be obtained for multiple values of k? Explain your answer.

■ Yes. For example if $k = 7$ then we get the same star as for $k = 5$, just 'drawn' in the reverse order. (For 5 we have the point sequence $\{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0\}$, while for 7 we have $\{0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0\}$.)

(c) Which k-stars for k between 1 and 11 inclusive can be drawn without ever taking your pen off the paper?

■ The 1-star, the 5-star, the 7-star, and the 11-star. (Here the valid values of k are all relatively prime to 12; that is, the only factor they share with 12 is 1.)

(d) Now suppose 30 points are marked around a circle. Which k-stars for k between 1 and 29 inclusive can be drawn without ever taking your pen off the paper?

■ The 1-star, the 7-star, the 11-star, 13-star, the 17-star, the 19-star, the 23-star, and the 29-star. (Here the valid values of k are all relatively prime to 30.)

Like Question 3, success here required a careful reading of the question. Quite a few students drew just one triangle for the 4-star, ignoring step (3) in the list of instructions. Although many students said 'yes' in part (b), not many could explain why, while in (c) and (d) the fact that the valid values of* k *needed to be relatively prime to the number of points was missed by many students.

Question 6: 20 marks (All Years)

While doing a computing course, Carol investigates *8-bit* numbers. These are numbers written in *binary notation*; that is, every digit is written as either a 1 or a 0. 8-bit numbers have 8 digits in total, and always include the leading zeroes. For example, 00011001, 01111100, and 11100011 are all 8-bit numbers. (We never write our first two examples as 11001 or 1111100.)

If Carol writes an 8-bit number as $abcdefgh$, where a, b, c, d, e, f, g, and h are either 0 or 1, she can convert her number into decimal notation by using the following equation:

$$\text{(number in decimal notation)} = 2^7a + 2^6b + 2^5c + 2^4d + 2^3e + 2^2f + 2^1g + 2^0h.$$

Note that every 8-bit number has exactly one decimal equivalent, and vice versa.

(a) What is the decimal equivalent of the 8-bit number 11100100?

■ $2^7 \times 1 + 2^6 \times 1 + 2^5 \times 1 + 2^4 \times 0 + 2^3 \times 0 + 2^2 \times 1 + 2^1 \times 0 + 2^0 \times 0 = 228$.

(b) What is the 8-bit equivalent of the decimal number 104?

■ $104 = 2^6 + 2^5 + 2^3$, which means b, c, and e are 1 while the rest of the digits are 0. Thus the 8-bit number is 01101000.

(c) What is the decimal equivalent of the largest 8-bit number?

■ The largest 8-bit number has all its digits equal to 1. Thus $2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 255$.

One way of combining two 8-bit numbers is to use a *logic gate*, where each digit of the two numbers is combined using a particular rule to produce either a 0 or 1 in the new 8-bit number. Here are three such rules:

(1) AND: if both digits are 1, the result is 1, otherwise 0.

(2) OR: if both digits are 0, the result is 0, otherwise 1.

(3) XOR: if exactly one of the two digits is 1, the result is 1, otherwise 0.

For example, 01010111 AND 11100000 = 01000000 and 01010111 OR 11100000 = 11110111. The order we in which have our two numbers is not important: for example, 01010111 XOR 11100000 and 11100000 XOR 01010111 are both equal to 10110111.

The output of combining two 8-bit numbers with a logic gate is always another 8-bit number. This means that logic gates can be used multiple times to combine any number of 8-bit numbers into a single 8-bit number. Again the order of the inputs does not matter. For example, given four 8-bit numbers a, b, c, and d, the following combinations (**amongst others**) produce the same 8-bit number as a final output: (a OR b) AND (c OR d), (b OR a) AND (c OR d), (c OR d) AND (a OR B), and (d OR c) AND (b OR a).

Suppose Carol wishes to combine the 8-bit numbers 11001100, 10101010, 11100110, and 00001001 into one 8-bit number using a combination of logic gates.

(d) Carol first combines 11001100 and 10101010 with AND, then 11100110 and 00001001 with OR. If she combines the outputs of these two logic gates with XOR, what final output does she get?

■ 11001100 AND 10101010 = 10001000, while 11100110 OR 00001001 = 11101111. Then 10001000 XOR 11101111 = 01100111.

(e) If the output Carol wants is 01100110, provide a way she can join together any three of the logic gates described above with her four input numbers listed above to achieve this. (Each gate can be used multiple times or not at all, but exactly three gates are to be used.)

■ Let a = 11100110 AND 00001001 = 00000000 and b = 10101010 XOR 11001100 = 01100110. Then a XOR b = 00000000 XOR 01100110 = 01100110. [Several answers are valid here.]

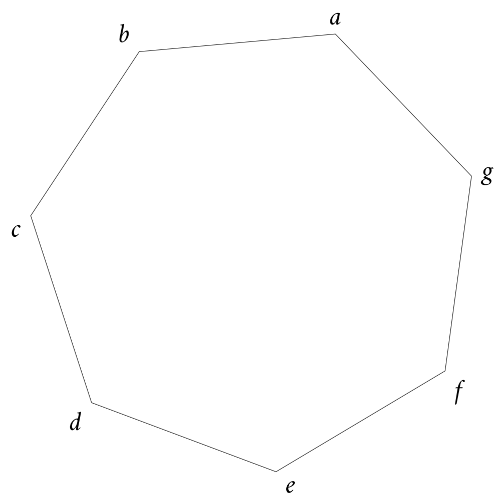
(f) Is the way you found in (e) the only way Carol could have reached 01100110 from her four chosen inputs? If so, briefly explain why this would be the case. If not, provide another way of joining together the logic gates.

■ There is more than one way Carol could combine her four chosen inputs to produce 01100110. For example, let a = 11100110 AND 00001001 = 00000000, then let b = a XOR 10101010 = 10101010. Finally b XOR 11000110 = 01100110 as desired. [Several answers are valid here, as long as the answer given is different from the answer given in (e).]

Results were mixed here. It's clear that many students had encountered binary numbers before, and for those students answering the first four parts of the question was relatively straightforward. Most students found the last two parts much harder however, and in the end only one student in the competition earned full marks in this question. Note that in many computing operations the value of every bit is important, so leaving off any leading 0s from an 8-bit number is generally considered bad practice.

Question 7: 10 marks (Years 10 and 11 only)

Consider a regular heptagon with side length 1 cm and with vertices labelled a, b, c, d, e, f, and g. Let T be the set of all triangles that can be constructed out of any three vertices of said heptagon, and let A be the subset of T containing every member of T with a as a vertex.



(a) How many triangles are in A?

> ■ We can construct the following triangles that have a as a vertex: $\triangle agf$, $\triangle age$, $\triangle agd$, $\triangle agc$, $\triangle agb$, $\triangle afe$, $\triangle afd$, $\triangle afc$, $\triangle afb$, $\triangle aed$, $\triangle aec$, $\triangle aeb$, $\triangle adc$, $\triangle adb$, and $\triangle acb$. There of 15 of these in total.

(b) How many triangles are in T?

> ■ From the 7 vertices in the heptagon, we can choose any 3 to make a triangle. Thus there are $\binom{7}{3} = \frac{7!}{3!4!} = 35$ possible triangles.

We say two triangles s and t are *congruent* if for each side of s there is a side of t with the same length (and vice versa), and for each angle of s there is an angle of t of the same size (and vice versa). If s and t are congruent triangles we can denote this in shorthand by saying $s \cong t$ (if s and t are not congruent we write $s \ncong t$ instead).

(c) How many triangles in A are not congruent to $\triangle agf$? (Recall that all members of A have a as a vertex.)

> ■ Both $\triangle acb$ and $\triangle agb$ are congruent to $\triangle agf$. No other triangle in B with a as a vertex has two sides of length 1 cm, thus there are 12 triangles in B with a as a vertex that are not congruent to $\triangle agf$. (All triangles are congruent to themselves.)

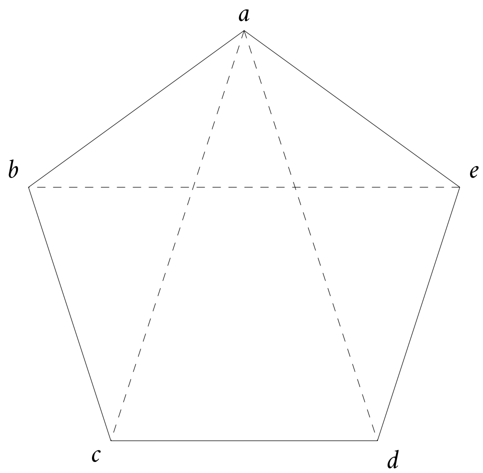
(d) Suppose M is any subset of A where for any pair of triangles $s, t \in M$ we have $s \ncong t$. What is the most number of triangles that can be in M?

> ■ Suppose $\triangle agf \in M$. We have $\triangle agf \ncong \triangle age$, so let $\triangle age \in M$ as well. Since $\triangle afd \ncong \triangle agf$ and $\triangle afd \ncong \triangle age$, we can have $\triangle agf \in M$ as well. Now $\triangle agd$ is not congruent to any of $\triangle agf$, $\triangle age$, or $\triangle afd$, so let $\triangle agd \in C$.
>
> The remaining triangles in A are now congruent to a member of M:
>
> $\triangle agf$**:** congruent to $\triangle acb$ and $\triangle agb$.
>
> $\triangle age$**:** congruent to $\triangle afe$, $\triangle afb$, $\triangle agc$, $\triangle adc$, and $\triangle adb$.
>
> $\triangle afd$**:** congruent to $\triangle aec$ and $\triangle afc$.
>
> $\triangle agd$**:** congruent to $\triangle aeb$ and $\triangle aed$.
>
> Thus the most number of triangles that can be in M is 4. (If we replace one of $\triangle agf$, $\triangle age$, $\triangle afd$, or $\triangle agd$ with a congruent triangle from A, this doesn't allow us to add any more triangles to M.)

This was by far and away the hardest question in the competition. Nobody scored full marks in this question, and correct answers to (d) in particular were very rare. Note there are multiple ways to answer (a) and (b); most students who gave correct answers used combinations to do so. The model answer presented in (a) is more verbose than that often given by students but was chosen for its ability to be understood for those students without any background in combinatorics.

Question 8: 10 marks (Year 11 only)

Consider a regular pentagon with side length 1 cm and with vertices labelled a, b, c, d, and e. (The pictured diagram is *not to scale*.)



(a) Find the area of the triangle $\triangle abe$.

■ The angle $\angle eab$ has size $(180 - 72) = 108°$. (72° is the external angle of a regular polygon.) If we have a point f that is halfway between b and e, we create a right angle triangle $\triangle afe$ with a hypotenuse of 1 cm. Thus the length of line $\overline{fe}$ is $\sin(54) = 0.809$ cm to three decimal places, and the length of line $\overline{af}$ is $\cos(54) = 0.587$ cm. Since $\overline{fe}$ is half the base of $\triangle abe$ it follows that the area of $\triangle abe$ is $0.809 \times 0.587 = 0.476$ cm to three decimal places.

(b) Find the area of the triangle $\triangle acd$.

■ $\triangle acd$ is also an isosceles triangle. By rotational symmetry from (a) the length of lines $\overline{ad}$ and $\overline{ac}$ are both 1.618 cm to three decimal places. Consider the point g halfway between c and d. The length of $\overline{ag}$ can found using Pythagoras' theorem: $\overline{ag} = \sqrt{1.618^2 - \frac{1}{2}^2} = 1.539$ cm to three decimal places. It follows that the area of $\triangle acd$ is 0.769 cm to three decimal places (as half the base is 0.5 cm).

Compared to Question 7, Year 11 students did relatively well here, and also did better comparatively to the equivalent question in last year's competition. There are of course multiple paths to a correct solution in each part. For example, the isosceles triangle area formula may be used to find a correct answer in both parts. Like in Question 7(a) we have chosen more verbose model solutions for the benefit of those with only a basic knowledge of trigonometry.