Tutorial 7 Sample Problems Solutions

These solutions are intended as a guide only. There are certainly other correct ways to do each problem. Additionally, in the solutions, some of the steps may not be spelled out in detail. If you have trouble understanding them, please see your lecturer or tutor.

Note also that you may possibly gain full marks for the assignment without writing down as much information as the solutions provide. However, you should always strive to explain clearly and succinctly what you are doing, which will involve using some words – not just bits of unrelated mathematics!!

1. *On a hot summers day in Dunedin the temperature (in degrees Celsius) is estimated to be* $y = f(x) = 30 - (x-5)^2 = -x^2 + 10x + 5$, *where* x *is the number of hours that have elapsed since 8am (valid for* $0 \le x \le 10$*).*

    (a) *Find the average rate of change in temperature from 8am to 10am (*$x = 0$ *to* $x = 2$*) and from 10am to 1 pm.*

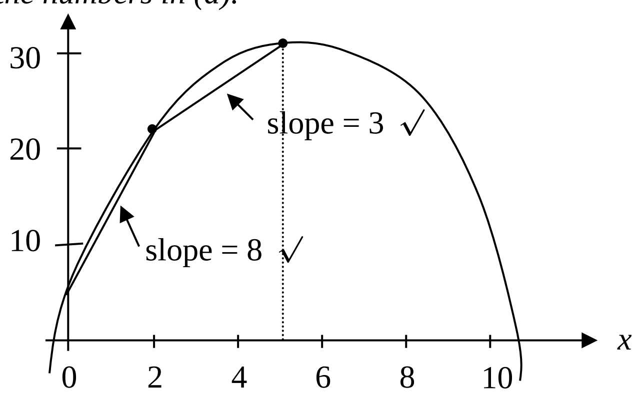
    The average rate of change from 8am to 10am, that is $x = 0$ to $x = 2$, is given by

    $$\frac{\Delta y}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{21 - 5}{2} = 8 \text{ (degrees per hour).} \ \checkmark$$

    and from 10am to 1pm, that is $x = 2$ to $x = 5$, is given by

    $$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(2)}{5 - 2} = \frac{30 - 21}{3} = 3 \text{ (degrees per hour)} \ \checkmark$$

    (b) *Roughly sketch the graph of* y *and draw some lines on it whose slopes represent the numbers in (a).*



2. *When Sally goes down a long slide the distance she has gone after x seconds is given by $y = f(x) = 3x^2$.*

(a) *Compute her average speed between $x = 1$ and $x = 1.1$ and between $x = 1$ and $x = 1.01$. Guess her instantaneous speed at $x = 1$.*

Her average speed between $x = 1$ and $x = 1.1$ is given by

$$\frac{f(1.1)-f(1)}{1.1-1} = \frac{3(1.1)^2-3(1)}{0.1} = 6.3 \text{ m/sec} \checkmark$$

and between $x = 1$ and $x = 1.01$ is given by

$$\frac{f(1.01)-f(1)}{1.01-1} = \frac{3(1.01)^2-3(1)}{0.01} = 6.03 \text{ m/sec} \checkmark$$

We guess her instantaneous speed at $x = 1$ is 6m/sec.

(b) *Calculate $\frac{f(1+h)-f(1)}{h}$. What is the slope of $f(x)$ at $x = 1$?*

$$\frac{f(1+h)-f(1)}{1+h-1} = \frac{3(1+h)^2-3(1)}{h} = \frac{3(1+2h+h^2)-3}{h}$$

$$= \frac{6h+3h^2}{h} = \frac{3h(2+h)}{h} = 3(2+h) = 6 + 3h. \checkmark$$

As $h \to 0$, $6 + 3h \to 6$ so the slope of $f(x)$ at $x = 1$ is 6. ✓

(c) *Find $\frac{dy}{dx}$ at $x = 1$.*

$\frac{dy}{dx} = 6x$, ✓ so when $x = 1$, $\frac{dy}{dx} = 6$. ✓

3. (a) *If $f(x) = 2x^3 - 3x$, find $f'(3)$.*

$f'(x) = 6x^2 - 3$, ✓ so $f'(3) = 54 - 3 = 51$. ✓

(b) *If $f(x) = 3x^2 - \frac{2}{x}$, find $f'(2)$.*

$f(x) = 3x^2 - 2x^{-1} \Rightarrow f'(x) = 6x + 2x^{-2} = 6x + \frac{2}{x^2}$.✓

$\Rightarrow f'(2) = 12 + \frac{2}{4} = 12\frac{1}{2}$ ✓

(c) *If $y = 5\sqrt{x} + 3x^{10}$, find $\frac{dy}{dx}$ at $x = 1$.*

$y = 5\sqrt{x} + 3x^{10} = 5x^{1/2} + 3x^{10} \Rightarrow$

$\frac{dy}{dx} = \frac{5}{2}x^{-\frac{1}{2}} + 30x^9 \Rightarrow$ At x = 1, $\frac{dy}{dx} = \frac{5}{2} + 30 = 32\frac{1}{2}$. ✓✓

4. *A ball is thrown vertically upwards from a point above the ground so its height h (in metres) after t seconds is given by* $h = -t^2 + 6t + 7$.

(a) *What height was it thrown from?*

$t = 0 \Rightarrow h = 7$. √

(b) *Give an expression for its velocity v at time t.*

$v = \frac{dh}{dt} = -2t + 6$ (m/sec). √

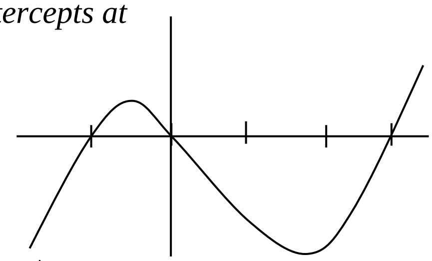
(c) *Find the time at which it reaches its greatest height (use calculus) and the height at this time.*

Maximum height occurs when $v = 0$. Now $v = 0 \Rightarrow -2t + 6 = 0 \Rightarrow$ $t = 3$ secs. √□□ When $t = 3$, $h = -9 + 18 + 7 = 16$ (metres). √

(d) *Find the time when it hits the ground and the ball speed at that time.*

$h = 0 \Rightarrow t^2 - 6t - 7 = 0 \Rightarrow (t - 7)(t + 1) = 0 \Rightarrow$ $t = -1$ (discard) or $t = 7$. √ When $t = 7$, $v = -14 + 6 = -8$ m/sec √□(so its speed is 8 m/sec downwards).

5. *Let* $f(x) = x(x + 1)(x - 3) = x(x^2 - 2x - 3)$ *(which has x intercepts at* -1, 0 *and* 3 *and the general shape shown).*



(a) *Find the equation of the tangent to the curve at* $x = 2$.

$f(x) = x(x^2 - 2x - 3) = x^3 - 2x^2 - 3x$

so $f'(x) = 3x^2 - 4x - 3$ and $f'(2) = 12 - 8 - 3 = 1$. √

When $x = 2$, $y = f(2) = 8 - 8 - 6 = -6$. √

Therefore the tangent at $x = 2$ has slope $m = 1$ and passes through the point $(2, -6)$ so has equation $y + 6 = 1(x - 2)$ √ or $y = x - 8$. (√)

(b) *What are the x-coordinates of the points where the slope of the curve is 4?*

√

$f'(x) = 4 \Rightarrow 3x^2 - 4x - 3 = 4 \Rightarrow 3x^2 - 4x - 7 = 0 \Rightarrow$

$(3x - 7)(x + 1) = 0 \Rightarrow x = -1$ √□□or $x = \frac{7}{3}$. √