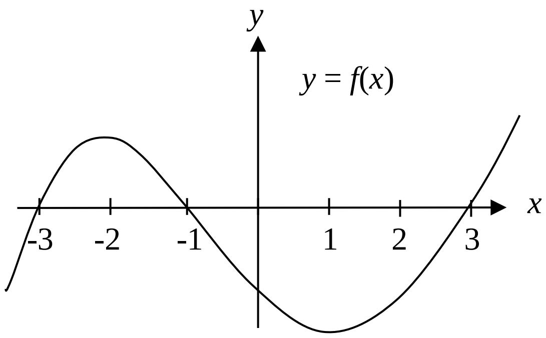
Tutorial 8 Sample Problem solutions

These solutions are intended as a guide only. There are certainly other correct ways to do each problem. Additionally, in the solutions, some of the steps may not be spelled out in detail. If you have trouble understanding them, please see your lecturer or tutor.

Note also that you may possibly gain full marks for the assignment without writing down as much information as the solutions provide. However, you should always strive to explain clearly and succinctly what you are doing, which will involve using some words – not just bits of unrelated mathematics!!

1. *For the graph of $f(x)$ shown, decide for what value(s) of x*

   (a) *the derivative $f'(x) > 0$*
   $x < -2$ or $x > 1$ (√√)
   (b) both $f(x) > 0$ and $f'(x) < 0$
   $-2 < x < -1$. (√)
   (c) $f(x) < 0$ and $f'(x) = 0$
   $x = 1$ (√).

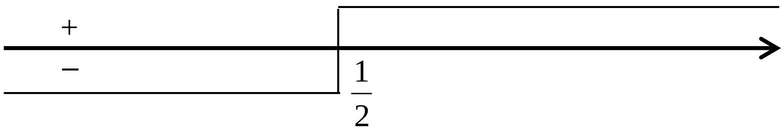


2. *For each function $f(x)$ below, find its derivative and stationary value(s). Draw a sign diagram for the derivative and hence find intervals on which the function is increasing and on which it is decreasing and decide the nature of the stationary points (whether a local maximum or minimum).*

   (a) $f(x) = x^2 - x + 1$

$f(x) = x^2 - x + 1 \Rightarrow f'(x) = 2x - 1$ (√), so $f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Now $f(½) = ¾$, so the stationary value is (½, ¾) (√). Further, $f'(x) > 0$ when $x > ½$, and $f'(x) < 0$ when $x < ½$, so the sign diagram for $f'(x)$ is:



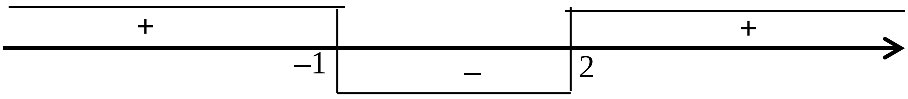
(√)

That is, $f(x)$ is decreasing when $x < ½$ and increasing when $x > ½$. (√)
The stationary point is therefore a minimum.

(b) $f(x) = 2x^3 - 3x^2 - 12x$.

$f(x) = 2x^3 - 3x^2 - 12x \Rightarrow f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$ (√),
so $f'(x) = 0$ where $x = 2$ or $x = -1$.

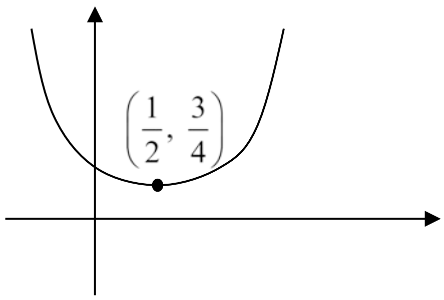
Now $f(-1) = 7$ and $f(2) = -20$, so the stationary values are $(-1, 7)$ and $(2, -20)$. Further $f'(x)$ is a quadratic facing up ($\cup$), so is negative between its roots and positive elsewhere, so the sign diagram for $f'(x)$ is:

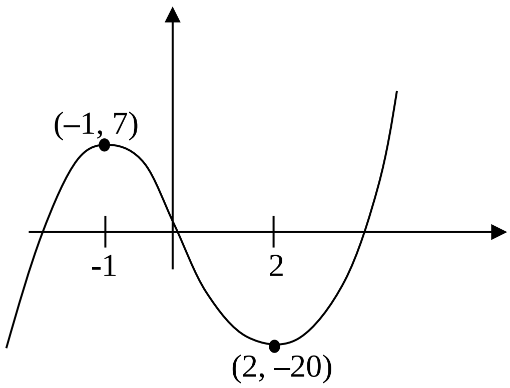


(√)

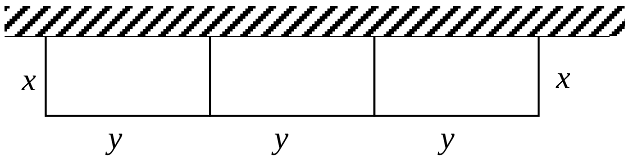
That is, $f(x)$ is increasing when $x < -1$ or $x > 2$ and decreasing when $-1 < x < 2$. (√) From the sign diagram we see $(-1, 7)$ is a local maximum and $(2, -20)$ is a local minimum.(√)

The graphs (not asked for) are shown below.





3. *A farmer wishes to make three identical rectangular enclosures, as shown using an existing wall as one boundary. If he has 200 metres of fencing, what should the dimensions of each enclosure be if the total area is to be maximized?*



The total area $A = 3xy$, where $4x + 3y = 200$ (since the total fencing available is 200 metres. Thus $3y = 200 - 4x$, so $A = 3yx = (200 - 4x)x = 200x - 4x^2$. (√)

Now $\frac{dA}{dx} = 200 - 8x$ (√), so $\frac{dA}{dx} = 0 \Rightarrow 8x = 200 \Rightarrow x = 25$ and $y = 100/3$ (√) (since $3y = 200 - 4x = 200 - 100 = 100$). This stationary value is clearly a maximum (√) (since A is a quadratic facing downward $\cap$ or by using the sign diagram for $\frac{dA}{dx}$), so to maximize A each enclosure should be

$25 \times 100/3$ (square metres).

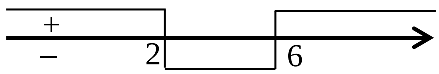
4. *In a far off place called Keyland the price of a share in Foreshore Enterprises is predicted to vary according to the formula*

$$P(t) = t^3 - 12t^2 + 36t + 10$$

where $P(t)$ is the price in dollars and $t \geq 0$ is the time in months since Jan 1, 2010.

(a) *Determine the intervals when the share price is (i) increasing (ii) decreasing.*

$P'(t) = 3t^2 - 24t + 36$ (√) $= 3(t^2 - 8t + 12) = 3(t-2)(t-6)$ which is a quadratic facing up (∪), so positive except between the roots 2 and 6. Therefore the share price $P(t)$ is (i) increasing when $t < 2$ and $t > 6$ (√) and (ii) decreasing when $2 < t < 6$. (√) The sign diagram (not required) :



(b) *Determine the times at which the share price is a local (i) minimum (ii) maximum, and give the corresponding price of a share.*

$P'(t) = 0 \Rightarrow (t-2)(t-6) = 0 \Rightarrow t = 2$ or $t = 6$ (√), so the stationary values are $(2, P(2)) = (2, 42)$ and $(6, P(6)) = (6, 10)$.

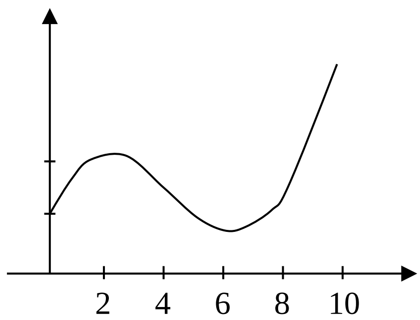
From the sign diagram for $P'(t)$ above (or the general shape of cubic with coefficient of t^3 positive) we see $P(t)$ has (i) a local minimum (\$10) at t = 6 (√) and (ii) a local maximum (\$42) at $t = 2$. (√)

(c) *What is predicted to be the maximum price of a share during the first (i) 3 months (ii) 9 months of 2010?*

To find the (absolute) maximum share price on a restricted domain we must also check the share price at the end points.

(i) On [0, 3] the maximum \$42 occurs at the local maximum when $t = 2$ (√), but

(ii) on [0, 9], since $P(9) = \$91$, the maximum occurs at the endpoint $t = 9$. (√)



5. *In a two party political system the percentage of voters who support the opposition parties has been predicted to be*

$$P(t) = t^4 - 4t^3 + 4t^2 + 45,$$

Where $t \geq 0$ *is the time in years since the last election.*

(a) *Find* $P'(t)$ *and by factorising* $P'(t)$ *show that the stationary points occur when* $t = 0, 1, 2$. *What are the corresponding values of* $P(t)$*?*

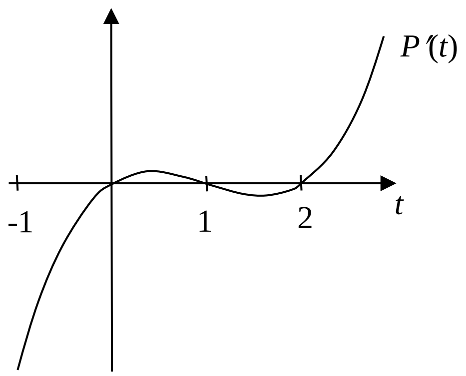
$P'(t) = 4t^3 - 12t^2 + 8t = 4t(t^2 - 3t + 2) = 4t(t - 1)(t - 2)$ (√)

Thus stationary values occur when $t = 0, 1, 2$.

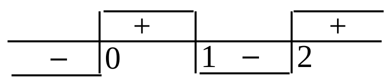
Further $P(0) = 45$, $P(1) = 46$ $P(2) = 45$. (√)

(b) *Draw the sign diagram for* $P'(t)$ *and so decide when* $P(t)$ *is a local maximum.*

$P'(t)$ is a cubic with $a = 4 > 0$ so has graph as shown and thus the corresponding sign diagram given.



Sign diagram



(√√)

Thus $P(t)$ has a local maximum at $t = 1$, since there the slope changes from + to – . (√)

(c) *Would it be better for the Government to call a snap election after 2 years or have the election as usual after 3 years?*

Best to call a snap election (√) because $P(2) = 45$ (a local minimum in fact so the opposition vote here is at its lowest) whereas $P(3) = 54$ so at $t = 3$ the opposition would win.